ANALYTICAL METHOD FOR ANALYSIS AND DESIGN OF CONTROL SYSTEMS

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Submitted in partial fulfillment of
the requirements for the degree of

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ABSTRACT

A new method for design of control systems in the time domain is presented. The adjustable parameters, the roots and the zeros of the system are evaluated simultaneously either by hand or by computer. The method is applicable to any system configuration and is able to handle a large number of adjustable parameters. The method is analytical, any degree of accuracy can be obtained.

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PART I ANALYSIS

CHAPTER I

THEORETICAL CONSIDERATION OF ALGEBRAIC EQUATIONS

1-1. Introduction.

The response of feedback control systems is governed by the complete set of the closed loop pole-zero configurations. Since the zeros are usually known, the only problem is the poles. For linear control systems, the problem of poles is conveyed to the roots of the characteristic equation which is an algebraic equation. Several methods have been developed to this end such as Guilemin's method[1], root-locus method and Mitrovic's method[2,3,4]. Guillemin's method starts the synthesis by choosing a set of pole-zero configurations from the consideration of specifications and then finds out the network needed to give this chosen pole-zero configuration. The root-locus method is essentially a graphic method for one variable which has linear relation with the coefficients of the characteristic equation. Mitrovic's method is also a graphic method but handles two variable coefficients. The method proposed in this paper is rather an analytic method and is able to handle any number of variables which are contained in the coefficients of the characteristic equation. The analysis is based on the root-coefficient relation of an algebraic equation, but a new technique of partitioning the equation is employed.

1-2. Root-coefficient Relationship.

For a n^{th} order algebraic equation, there are n relations among the roots and the coefficients of the equation. If all the coefficients are actually known numerically, then all the roots are fixed in such a way that

they satisfy the root-coefficient relations. Conversely, if a set of roots of the characteristic equation are chosen, then all the coefficients of the equation are fixed by those relations. However, if variables are contained in the coefficients, then the roots of the equation are varied in a definite relation to the variation of the variables. For example, if one coefficient of the characteristic equation is variable, then the relation between the roots and the variable coefficient can be plotted by root-locus method. But if this relationship is looked at from a different point of view, an analytical relation can be obtained. Consider a third order characteristic equation with one variable coefficient as shown in the following equation:

$$S^3 + B_2 S^2 + f_1(h) S + B_0 = 0 \tag{1-1}$$

where B_2 and B_0 are known numerically, $f_1(h)$ is a function of h which can be varied. Assume $-S_1$, $-S_2$ and $-S_3$ are the three roots of the equation, then the root-coefficient relations are as follows:

$$S_1 + S_2 + S_3 = B_2 \tag{1-2-1}$$

$$S_1 S_2 + S_2 S_3 + S_3 S_1 = f_1(h) \tag{1-2-2}$$

$$S_1 S_2 S_3 = B_0 \tag{1-2-3}$$

Since $f_1(h)$ is adjustable, then equation (1-2-2) can always be satisfied for any choice of the roots by varying $f_1(h)$ accordingly. However, equations (1-2-1) and (1-2-3) set constraints to the value of the roots. Namely, the three roots must satisfy the two equations (1-2-1) and (1-2-3).

If the three roots are treated as the variables of a set of simultaneous equations, then there are three unknowns but only two equations, consequently one root is arbitrary. This arbitrary root can be chosen independently if there is no restriction to the value of $f_1(h)$, and therefore this root can be treated as an independent variable in the root-coefficient relations. In the root-locus method, the independent variable is $f_1(h)$ while in this point of view, the independent variable is conveyed to one of the roots of the characteristic equation.

The above reasoning is based on the solution of simultaneous algebraic equations, i.e., S_1 , S_2 and S_3 are treated as independent variables. Since those variables are the roots of an algebraic equation, a complex root must be treated as two variables because of its conjugate root. If one variable of a pair of complex conjugated roots is expected to be treated as the independent variable, this variable cannot be chosen to be the root itself, but can be treated as follows. Assume $-S_1$ and $-S_2$ are a pair of complex conjugate roots of the equation (1-1), those two roots can be transformed to other two variables by the following transformations:

$$S_1 + S_2 = 2\zeta\omega_n \tag{1-3}$$

$$S_1 S_2 = \omega_n^2 \tag{1-4}$$

Where ζ and ω_n are known as the damping ratio and the natural frequency of the pair of complex conjugate roots $-S_1$ and $-S_2$. By this transformation, both ζ and ω_n can be treated as independent variables.

In equation (1-1), if $f_1(h)$ was also a fixed number, then all roots were fixed numbers by equations (1-2) and no root was arbitrary. The arbitrary roots arise from the introduction of variables to the coefficients. If one variable is introduced to the coefficients, it has been

shown one root is arbitrary. For the same reason, if two variables are introduced to the coefficient of a n^{th} order characteristic equation, then two roots are arbitrary, and for m variables to the coefficients, m roots are arbitrary. The root-locus method is the case of one variable while Mitrovic's method is the case of two variables, but both methods are quite different ways of approach. It is because of this way of looking at the root-coefficient relations, an analytic method can be derived and is capable of handling any number of variables contained in the coefficients.

1-3. Partition of a Characteristic Equation.

If adjustable parameters are introduced to the coefficients of a characteristic equation, then this equation can be partitioned into two equations. One of them is formed by the arbitrary roots, the other is formed by the constrained roots. The constrained roots here are defined as those roots other than arbitrary.

Consider a 5th order characteristic equation

$$S^5 + B_4 S^4 + B_3 S^3 + B_2 S^2 + B_1 S + B_0 = 0 \qquad (1\text{-}5\text{-}1)$$

Assume $-S_1$, $-S_2$, $-S_3$, $-S_4$ and $-S_5$ are the roots. The negative sign is for convenience. By expanding the equation

$$(S+S_1)(S+S_2)(S+S_3)(S+S_4)(S+S_5) = 0 \qquad (1\text{-}5\text{-}2)$$

Set the corresponding coefficients of (1-5-1) and (1-5-2) equal, then the root-coefficient relations are obtained as follows:

$$B_4 = S_1 + S_2 + S_3 + S_4 + S_5 \qquad (1\text{-}6\text{-}1)$$

$$B_3 = (S_1+S_2)(S_3+S_4) + S_1 S_2 + S_3 S_4 + (S_1+S_2+S_3+S_4)S_5 \qquad (1\text{-}6\text{-}2)$$

$$B_2 = S_3S_4(S_1+S_2) + (S_3+S_4)S_1S_2 + \left[(S_1+S_2)(S_3+S_4) + S_1S_2 + S_3S_4\right]S_5 \quad (1\text{-}6\text{-}3)$$

$$B_1 = S_1S_2S_3S_4 + \left[(S_1+S_2)S_3S_4 + S_1S_2(S_3+S_4)\right]S_5 \quad (1\text{-}6\text{-}4)$$

$$B_0 = S_1S_2S_3S_4S_5 \quad (1\text{-}6\text{-}5)$$

Assume the coefficients B_1 and B_2 are adjustable, then equation (1-5-1) can be written as follows:

$$S^5 + B_4S^4 + B_3S^3 + f_2(h_1,h_2)S^2 + f_1(h_1,h_2)S + B_0 = 0 \quad (1\text{-}7)$$

Where $f_1(h_1, h_2)$ and $f_2(h_1, h_2)$ are functions of h_1 and h_2 which are adjustable parameters. B_0, B_3 and B_4 are assumed given fixed values. A terminology, "controlled characteristic equation" is given to equation (1-7) on the reason that its roots are controllable by the adjustable parameters. Since there are two adjustable parameters introduced to the coefficients, two roots can be chosen arbitrarily. Let those two arbitrarily chosen roots be ($-S_1$) and ($-S_2$), then the other three roots ($-S_3$), ($-S_4$) and ($-S_5$) are not arbitrary but constrained to the fixed coefficients (B_0, B_3 and B_4) and the arbitrarily chosen roots as shown in the last section. By separating the arbitrary roots and the constrained roots, equations (1-6-1), (1-6-2) and (1-6-5) can be arranged as follows:

From (1-6-1)

$$S_3 + S_4 + S_5 = B_4 - (S_1 + S_2) \quad (1\text{-}8)$$

From equation (1-6-2)

$$B_3 = (S_3 + S_4 + S_5)(S_1+S_2) + (S_3S_4 + S_4S_5 + S_5S_3) + S_1S_2 \quad (1\text{-}9)$$

From equation (1-6-5)

$$B_o = (S_1 S_2)(S_3 S_4 S_5) \tag{1-10}$$

It can be recognized that ($S_3 + S_4 + S_5$), ($S_3 S_4 + S_4 S_5 + S_5 S_3$) and ($S_3 S_4 S_5$) are the coefficients of a third order algebraic equation which has roots of $-S_3$, $-S_4$ and $-S_5$. Define

$$C_o \triangleq S_3 S_4 S_5 \tag{1-11-1}$$

$$C_1 \triangleq S_3 S_4 + S_4 S_5 + S_5 S_3 \tag{1-11-2}$$

$$C_2 \triangleq S_3 + S_4 + S_5 \tag{1-11-3}$$

Then the equation of the constrained roots is formed by

$$S^3 + C_2 S^2 + C_1 S + C_o = 0 \tag{1-12}$$

It must be noticed that equations (1-12) has the roots $-S_3$, $-S_4$ and $-S_5$ Substitute the definitions of (1-11) into (1-8), (1-9) and (1-10) and manipulating, one obtains

$$C_2 = B_4 - (S_1 + S_2) \tag{1-12-1}$$

$$C_1 = B_3 - C_2(S_1 + S_2) - S_1 S_2 \tag{1-12-2}$$

$$C_o = B_o / S_1 S_2 \tag{1-12-3}$$

Consider equations (1-12), C_o , C_1 and C_2 are functions of the arbitrary roots and the fixed coefficients B_o , B_3 and B_4 of the controlled characteristic equation (1-7). Therefore the roots of equation (1-12) are also functions of the arbitrary roots $-S_1$ and $-S_2$ and the fixed coefficients. Assume there are no restrictions to the variable coefficients

$f_1(h_1, h_2)$ and $f_2(h_1, h_2)$ then the controlled characteristic equation (1-7) is partitioned into the following equation

$$(S+S_1)(S+S_2)(S^3+C_2S^2+C_1S+C_0)=0$$

because both equations have the same roots. As C_0, C_1 and C_2 are functions of S_1 and S_2, then the five roots of the original characteristic equation (1-7) are completely determined by equation (1-12). Equation (1-12) is a third order algebraic equation and the coefficients are expressed analytically by (1-12-1), (1-12-2) and (1-12-3). Then the original 5th order characteristic equation is now reduced to a third order equation and by this reason, a terminology "REDUCED CHARACTERISTIC EQUATION" is defined for equation (1-12).

In the above arguments it is assumed that the variables h_1 and h_2 are adjusted such that two arbitrary roots are chosen. Therefore, the variables h_1 and h_2, which are the actual physical adjustable parameters of the system, must be determined. From equation (1-6-3), and (1-6-4) replace B_2 by $f_2(h_1, h_2)$ and B_1 by $f_1(h_1, h_2)$ since they are assumed adjustable, and manipulating, one obtains

$$f_2(h_1, h_2) = C_0 + C_1(S_1+S_2) + C_2S_1S_2 \qquad (1\text{-}14\text{-}1)$$

$$f_1(h_1, h_2) = C_0(S_1+S_2) + C_1S_1S_2 \qquad (1\text{-}14\text{-}2)$$

Substitute C_0, C_1 and C_2 from equations (1-12) into (1-14), obtain

$$f_2(h_1,h_2) = B_0/S_1S_2 + B_3(S_1+S_2) - B_4(S_1+S_2)^2 + B_4S_1S_2 + (S_1+S_2)^3 - 2S_1S_2(S_1+S_2) \qquad (1\text{-}15\text{-}1)$$

$$f_1(h_1,h_2) = B_0(S_1+S_2)/S_1S_2 + B_3S_1S_2 - B_4S_1S_2(S_1+S_2) + S_1S_2(S_1+S_2)^2 - (S_1S_2)^2 \qquad (1\text{-}15\text{-}2)$$

It can be seen from equations (1-15) that $f_1(h_1, h_2)$ and $f_2(h_1, h_2)$ are expressed analytically as a function of the arbitrary roots and the fixed

coefficients B_0 , B_3 and B_4 . And f_1, f_2 also can be expressed explicitly as functions of the arbitrary roots. Therefore if a set of roots has been chosen from the consideration of the reduced characteristic equation, then the physical adjustable parameters are readily determined by equations (1-15) or (1-14).

By this partitioning process it can be seen easily that the original fice root-coefficient relations of (1-6) are transformed into other five relations which are equations (1-12) and (1-14) or (1-13) and (1-15). For convenience (1-12) and (1-14) are repeated here:

$$
\begin{aligned}
C_2 &= B_4 - (S_1 + S_2) \\
C_1 &= B_3 - C_2(S_1 + S_2) - S_1 S_2 \\
C_0 &= B_0 / S_1 S_2 \\
f_1 &= C_0 + C_1(S_1 + S_2) + C_2 S_1 S_2 \\
f_2 &= C_0(S_1 + S_2) + C_1 S_1 S_2
\end{aligned}
\tag{1-16}
$$

In actual numerical computations, it is simpler to use equation (1-16) instead of (1-12) and (1-15), because of the progressive nature of the root-coefficient relations of algebraic equations. Namely, compute C_0 and C_2 first, then using these calculated data to compute C_1 and then f_1 and f_2 .

In general, the n root-coefficient relations of a nth order characteristic equation which has variables in coefficients can be arranged in any form as one wishes. The process derived here has certain significances in the design of control systems. First of all, the order of the characteristic equations which determines all the roots of the system is reduced by the number of the variable parameters. Secondary, the physical adjustable

parameters are expressed directly by the arbitrarily chosen roots. The stability and response of the system is determined completely by the reduced characteristic equation while the adjustable parameters are determined directly by the arbitrary roots. Since equations (1-16) are algebraic equations and if no singularity is involved, then solutions of h_1 and h_2 which actually determine the physical adjustable parameters always exist. Although this process does not guarantee the adjustable parameter obtained in this procedure to be always realizable, the realizable region can be found. In the design of a control system, this is a logical procedure, namely the pole-zero configuration of the closed loop system is chosen first from the specifications, and then compute compensator parameters directly from the chosen pole-zero configurations. For every choice of pole-zero configuration, a corresponding set of values of the compensator are obtained. This partitioning process effectively transforms the desired pole-zero configuration to be the independent variables of the design procedure. Moreover, most of the designs are based on a pair of dominant complex conjugate roots, then two of the arbitrary roots can be chosen (not necessary) as the dominant roots. The adjustable parameters of the compensator therefore can be expressed explicitly as functions of the dominant roots if the dominant roots are defined.

In this process, there are no limitations. For a control system of any order, any number of adjustable parameters and any type of compensator; the same technique can be applied.

1-4. General Procedures of Partitioning a Controlled Characteristic Equation.

The example in the last section assumed that the adjustable parameters are contained in two coefficients. Now consider a cascade single section

R-C compensator as shown in the block diagram of Fig. 1-1, in which the compensator parameters enter all the coefficients. The characteristic equation of the compensated system is

$$s^4 + (B_2 + p)s^3 + (B_1 + B_2 p)s^2 + (B_1 p + \alpha B_0)s + pB_0 = 0 \qquad (1\text{-}17)$$

where $B_0 = K$, $B_1 = ab$, $B_2 = a + b$

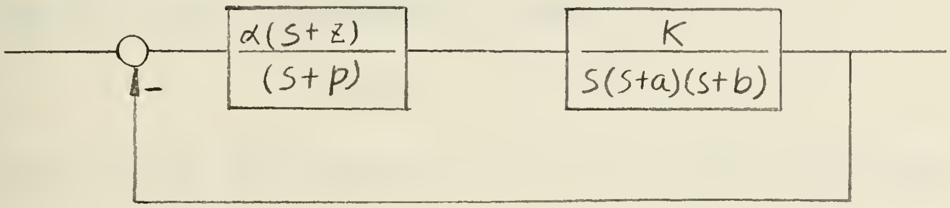


Fig. 1-1 Single section cascade compensator.

$\alpha = p/z$; K, a and b are know numerically.

Assume $-S_1$, $-S_2$, $-S_3$ and $-S_4$ are the roots of (1-17). By expanding the equation

$$(s + S_1)(s + S_2)(s + S_3)(s + S_4) = 0 \qquad (1\text{-}18)$$

and setting the corresponding coefficients of (1-17) and (1-18) equal, one obtains the following four relations:

$$B_2 + p = S_1 + S_2 + S_3 + S_4 \qquad (1\text{-}19\text{-}1)$$

$$B_1 + B_2 p = (S_1 + S_2)(S_3 + S_4) + S_1 S_2 + S_4 S_3 \qquad (1\text{-}19\text{-}2)$$

$$B_1 p + \alpha B_0 = S_1 S_2 (S_3 + S_4) + S_3 S_4 (S_1 + S_2) \qquad (1\text{-}19\text{-}3)$$

$$pB_0 = S_1 S_2 S_3 S_4 \qquad (1\text{-}19\text{-}4)$$

Since there are assumed two adjustable parameters α and p of the compensator, there are two arbitrary roots. Assume $-S_1$, and $-S_2$ are the

arbitrary roots, then ($S_3 + S_4$) and ($S_3 S_4$) are the coefficients of the reduced characteristic equation. Define

$$C_1 \triangleq S_3 + S_4 \tag{1-20-1}$$

$$C_0 \triangleq S_3 S_4 \tag{1-20-2}$$

then the reduced characteristic equation is

$$S^2 + C_1 S + C_0 = 0 \tag{1-21}$$

Substitute the definitions of (1-20) into (1-19), one obtains

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ (S_1+S_2) & 1 & -B_2 & 0 \\ S_1S_2 & (S_1+S_2) & -B_1 & -B_0 \\ 0 & S_1S_2 & -B_0 & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ C_0 \\ p \\ \alpha \end{bmatrix} = \begin{bmatrix} B_2-(S_1+S_2) \\ B_1 - S_1S_2 \\ 0 \\ 0 \end{bmatrix} \tag{1-22}$$

Equations (1-22) are the partitioned four root-coefficients relations in which C_1 and C_0 define the reduced characteristic equation while p and α are the compensator parameters. These four dependent variables (C_1, C_0, p and α) can be solved in terms of the arbitrary roots and the fixed plant parameters (B_0 , B_1 and B_2). For numerical computation purposes, only one dependent variable need be expressed explicitly as a function of the arbitrary roots, the others are computed readily by the previous calculations. Here C_0 is solved from (1-22) as shown in the following equation

$$C_0 = \frac{1}{\Delta}\left[S_1S_2 - (S_1+S_2)^2 - B_1 + B_2(S_1+S_2) \right] \tag{1-23-1}$$

where

$$\Delta = \left[B_2 - (S_1+S_2) \right] \frac{S_1S_2}{B_0} - 1 \tag{1-23-2}$$

Now consider another example which introduces three adjustable parameters. Fig. 1-2 is a fourth order system with first and second derivative feedback. The parameters h_1, h_2 and K (= K_1K_2) are assumed

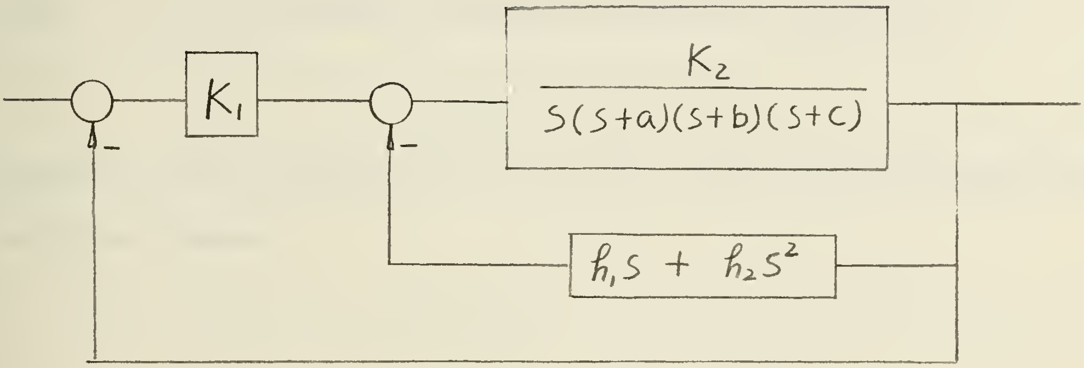


Fig. 1-2 Pure derivative feedback K, h_1 and h_2 are variables where $K = K_1K_2$

adjustable. The characteristic equation is

$$S^4 + B_3 S^3 + (B_2 + K_2 h_2) S^2 + (B_1 + K_2 h_1) S + K = 0 \tag{1-24}$$

where $B_3 = a + b + c$, $B_2 = ab + bc + ca$, and $B_1 = abc$. Assume the roots are $-S_1$, $-S_2$, $-S_3$ and $-S_4$ then the root coefficients relations are as follows:

$$B_3 = S_1 + S_2 + S_3 + S_4 \tag{1-25-1}$$

$$B_2 + K_2 h_2 = (S_1 + S_2) S_3 + S_1 S_2 + (S_1 + S_2 + S_3) S_4 \tag{1-25-2}$$

$$B_1 + K_2 h_1 = S_1 S_2 S_3 + S_4 [S_1 S_2 + S_3 (S_1 + S_2)] \tag{1-25-3}$$

$$K = S_1 S_2 S_3 S_4 \tag{1-25-4}$$

Since there are three adjustable parameters (h_1, h_2 and K), three roots are arbitrary. Let these arbitrary roots be $-S_1$, $-S_2$, and $-S_3$. The reduced characteristic equation is of order one and its roots is $-S_4$. Define $S_4 = C_o$, then the reduced characteristic equation is

$$S + C_o = 0 \tag{1-26}$$

Substitute the definition of $S_4 = C_o$ into (1-25), the root-coefficients relations become

$$C_o = B_3 - (S_1 + S_2 + S_3) \tag{1-27-1}$$

$$K_2 h_2 = (S_1 + S_2) S_3 + S_1 S_2 + C_o (S_1 + S_2 + S_3) - B_2 \tag{1-27-2}$$

$$K_2 h_1 = S_1 S_2 S_3 + C_o [S_1 S_2 + (S_1 + S_2) S_3] \tag{1-27-3}$$

$$K = S_1 S_2 S_3 C_o \tag{1-27-4}$$

Equations (1-27) are the partitioned root-coefficient relations in which (1-27-1) determine the system response while the other three equations determine the adjustable parameters so that to give the roots determined by (1-27-1).

From the three examples illustrated above, it can be seen that this partitioning process can be applied to systems of any order, any number of adjustable parameters and any system configurations. The important concept of this process is to separate the roots of the controlled characteristic equation into two parts: The arbitrary roots and the constrained roots. In the n root-coefficient relations, the arbitrary roots are treated as independent variables while the constrained roots and the adjustable parameters are the dependent variables. The constrained roots are obtained

from the solution of the reduced characteristic equation while the adjustable parameters are calculated from the arbitrary roots directly. The general procedures are summarized as follows:

(1) Formulate the controlled characteristic equation*. Set the coefficients equal to the corresponding root-coefficient.

(2) Define the arbitrary roots+ and the coefficients of the reduced characteristic equation (C_o, C_1...). Substitute C_o, C_1... into the equations in (1).

(3) Solve for the adjustable parameters and C's in terms of the arbitrary roots.

1-5. Summary.

This partitioning process effectively transforms the independent variables from the adjustable parameters to the arbitrary roots of the characteristic equation. As the adjustable parameters (compensator) are determined from the desired closed loop pole-zero configuration, a design procedure for control systems based on the time domain can be derived. The partitioning procedures presented in the last section are applicable to all system configurations. For specific cases of a nth order system, general formulas can be derived. For the cases that some of the coefficients themselves

*In discontinuous operation (2), the controlled characteristic equations may be different for different modes. As far as this step is concerned, each mode must be treated independently.

+ The number of arbitrary roots is equal to the number of the independent adjustable parameters. The independent adjustable parameters may be less than the compensator parameters because of the specifications and other restrictions. For details, it is discussed in Chapter IV.

are treated as variables, general expressions from the partitioning process have been derived as shown in Table 1-1 to Table 1-6. For other cases, they are shown in the subsequent Chapters. In order to illustrate the applicability, the following examples are presented.

Example: Given a characteristic equation

$$S^5 + 16S^4 + 128S^3 + 520S^2 + 1300S + 3000 = 0 \tag{1-28}$$

which is unstable. Assume a network can be devised such that the coefficients of the first, second and third derivative terms can be adjusted. Then the controlled characteristic equation is

$$S^5 + 16S^4 + f_3 S^3 + f_2 S^2 + f_1 S + 3000 = 0 \tag{1-29}$$

where f_1 , f_2 and f_3 are variables. From Table 1-5, the coefficients of the reduced characteristic equation are:

$$C_1 = B_4 - (S_1 + S_2 + S_3) \tag{1-30-1}$$

$$C_0 = B_0 / S_1 S_2 S_3 \tag{1-30-2}$$

The functions of the variable coefficients are:

$$f_1 = C_0 (S_1 S_2 + S_2 S_3 + S_3 S_1) + C_1 S_1 S_2 S_3 \tag{1-30-3}$$

$$f_2 = C_0 (S_1 + S_2 + S_3) + C_1 (S_1 S_2 + S_2 S_3 + S_3 S_1) + S_1 S_2 S_3 \tag{1-30-4}$$

$$f_3 = C_0 + C_1 (S_1 + S_2 + S_3) + (S_1 S_2 + S_2 S_3 + S_3 S_1) \tag{1-30-5}$$

where ($-S_1$), ($-S_2$) and ($-S_3$) are the arbitrarily chosen roots. Since equation (1-29) is of the 5th order, one root must be real. Assume

this real root is ($-S_3$). For convenience let ($-S_1$) and ($-S_2$) be the dominant complex roots and express them by the following notations:

$$S_1 + S_2 = 2\rho\omega_n \tag{1-31-1}$$

$$S_1 S_2 = \omega_n^2 \tag{1-31-2}$$

Substitute (1-31-1), (1-31-2), $B_4 = 16$, and $B_o = 3000$ into the equations (1-30), one obtains:

$$C_1 = 16 - (2\rho\omega_n + S_3) \tag{1-32-1}$$

$$C_o = 3000/\omega_n^2 S_3 \tag{1-32-2}$$

$$f_1 = C_o(\omega_n^2 + 2\rho\omega_n S_3) + C_1\omega_n^2 S_3 \tag{1-32-3}$$

$$f_2 = C_o(2\rho\omega_n + S_3) + C_1(\omega_n^2 + 2\rho\omega_n S_3) + \omega_n^2 S_3 \tag{1-32-4}$$

$$f_3 = C_o + C_1(2\rho\omega_n + S_3) + (\omega_n^2 + 2\rho\omega_n S_3) \tag{1-32-5}$$

The reduced characteristic equation is

$$S^2 + C_1 S + C_o = 0 \tag{1-33}$$

which determines all the roots of equation (1-29) and consequently determines the stability boundary. By putting all the three arbitrary roots on the left-half plane, then from (1-32-2) C_o is always positive, the condition of stability for the arbitrary roots therefore from (1-33) is

$$C_1 \geq 0$$

or

$$16 \geq 2\rho\omega_n + S_3 \tag{1-33}$$

The inequality (1-33) implies that the three arbitrary chosen roots are confined in a certain region if the system is to be stable. This region is plotted in Fig. 1-3. Notice $\zeta\omega_n$ is the real part of the complex roots. If S_1 and S_2 are complex the boundary is a line ($S = -8$). If S_1 and S_2 are real, the boundary is a point, ($S = -8 + j0$).

Assume the dominant roots ($-S_1$) and ($-S_2$) are desired such that all other roots must be to the left of the dominant roots, then the following inequalities must be satisfied

$$S_3 \geq \zeta\omega_n \tag{1-34-1}$$

and

$$C_1 \geq 2\zeta\omega_n \tag{1-34-2}$$

Substitute the boundary conditions of (1-34-1) and (1-34-2) into (1-32-1) one obtains

$$\zeta\omega_n \leq 16/5 \tag{1-35}$$

Equation (1-35) defines the dominant root region which is also shown in Fig. 1-3. Within this region, choose $\zeta = 0.5$, $\omega_n = 4$, $S_3 = 4$, then from (1-32-1) and (1-32-2)

$$C_1 = 8$$

$$C_0 = 47$$

The roots of (1-33) are: $-4 \pm j\,5.6$ ($\omega_n' = 6.85$, $\zeta' = 0.59$)

The root configuration for this choice of the arbitrary roots is also shown in Fig. 1-3.

The next step of the design is to calculate the variable coefficients. Substitute $\zeta = 0.5$, $\omega_n = 4$, $S_3 = 4$, $C_1 = 8$ and $C_0 = 47$ into equations (1-32-3), (1-32-4) and (1-32-5), one obtains

$$f_1 = 2015$$

$$f_2 = 696$$

$$f_3 = 143$$

For another choice of the arbitrary roots, the coefficients of the reduced characteristic equation and the variable coefficients are computed in the same way.

The above example assumed that three coefficients are variable. Other cases, such as a cascade compensator, feedback compensator, etc., can be carried out in the same procedure, since the expressions of the coefficients of the reduced characteristic equation and the adjustable parameter are obtained readily. The stable root region and dominant root region are important in design, the general methods for evaluating those two regions are discussed in the next chapter.

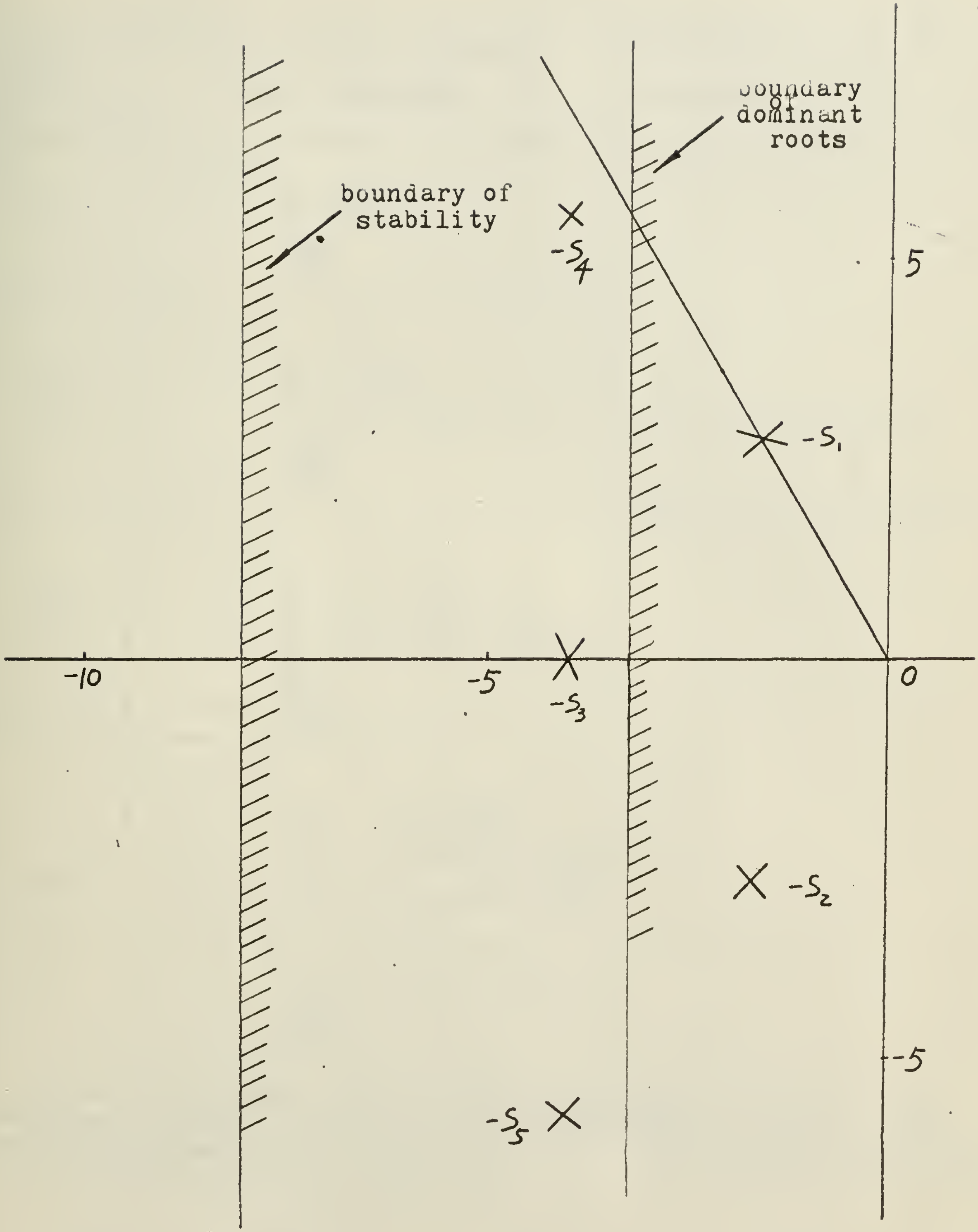


Fig. 1-3. Example for $s^5 + 10s^4 + f_3 s^3 + f_2 s^2 + f_1 s + 3000 = 0$

<table>
<tr><th>order of the equation</th><th>coeff. of the reduced chara. equation</th><th>functions of the variable coeff.</th></tr>
<tr><td>4th</td><td>$C_2 = 1$
$C_1 = B_3 - (S_1 + S_2)$
$C_0 = \frac{B_0}{S_1 S_2}$</td><td rowspan="4">$f_1 = C_0 (S_1 + S_2) + C_1 S_1 S_2$

$f_2 = C_0 + C_1 (S_1 + S_2) + C_2 S_1 S_2$</td></tr>
<tr><td>5th</td><td>$C_3 = 1$
$C_2 = B_4 - (S_1 + S_2)$
$C_1 = B_3 - C_2 (S_1 + S_2) - S_1 S_2$
$C_0 = \frac{B_0}{S_1 S_2}$</td></tr>
<tr><td>6th</td><td>$C_4 = 1$
$C_3 = B_5 - (S_1 + S_2)$
$C_2 = B_4 - C_3 (S_1 + S_2) - S_1 S_2$
$C_1 = B_3 - C_2 (S_1 + S_2) - C_3 S_1 S_2$
$C_0 = \frac{B_0}{S_1 S_2}$</td></tr>
<tr><td>nth</td><td>$C_{n-2} = 1$
$C_{n-3} = B_{n-1} - C_{n-2} (S_1 + S_2)$
$C_{n-4} = B_{n-2} - C_{n-3} (S_1 + S_2) - C_{n-2} S_1 S_2$
$C_{n-5} = B_{n-3} - C_{n-4} (S_1 + S_2) - C_{n-3} S_1 S_2$
$\vdots$
$C_1 = B_3 - C_2 (S_1 + S_2) - C_3 S_1 S_2$
$C_0 = \frac{B_0}{S_1 S_2}$</td></tr>
</table>

Table 1-1 Functions of the variable coefficients f_1 and f_2. The functions contained in this table are assumed only B_1 and B_2 are adjustable. $(-S_1)$ and $(-S_2)$ are the arbitrarily chosen roots and C's are the coefficients of the reduced characteristic equation.

$$S^n + C_{n-1} S^{n-1} + \cdots + C_1 S + C_0 = 0$$

order of the equation	coeff. of the reduced chara. equation	Functions of the variable coeff.
4th	$C_2 = 1$ $C_1 = [B_1 - C_0(S_1+S_2)]/S_1S_2$ $C_0 = B_0/S_1S_2$	
5th	$C_3 = 1$ $C_2 = B_4 - (S_1+S_2)$ $C_1 = [B_1 - C_0(S_1+S_2)]/S_1S_2$ $C_0 = B_0/S_1S_2$	$f_2 = C_0 + C_1(S_1+S_2) + C_2S_1S_2$ $f_3 = C_1 + C_2(S_1+S_2) + C_3S_1S_2$
6th	$C_4 = 1$ $C_3 = B_5 - (S_1+S_2)$ $C_2 = B_4 - C_3(S_1+S_2) - S_1S_2$ $C_1 = [B_1 - C_0(S_1+S_2)]/S_1S_2$	
nth	$C_{n-2} = 1$ $C_{n-3} = B_{n-1} - C_{n-2}(S_1+S_2)$ $C_{n-4} = B_{n-2} - C_{n-3}(S_1+S_2) - C_{n-2}S_1S_2$ $\vdots \quad \vdots \quad \vdots \quad \vdots$ $C_2 = B_4 - C_3(S_1+S_2) - C_4S_1S_2$ $C_1 = [B_1 - C_0(S_1+S_2)]/S_1S_2$ $C_0 = B_0/S_1S_2$	

Table 1-2. Functions of the variable coefficients f_2 and f_3. The functions contained in this table are assumed only B_2 and B_3 are adjustable. $(-S_1)$ and $(-S_2)$ are the arbitrarily chosen roots and C's are the coefficients of the reduced characteristic equation.

<table>
<tr><th>order of the equation</th><th>coeff. of the redeced chara. equation</th><th>Functions of the variable coeff.</th></tr>
<tr><td>4th</td><td>$C_2 = 1$
$C_1 = (B_2 - C_0 - S_1S_2)/(S_1+S_2)$
$C_0 = B_0/S_1S_2$</td><td rowspan="4">$f_1 = C_0(S_1+S_2) + C_1S_1S_2$

$f_3 = C_1 + C_2(S_1+S_2) + C_3S_1S_2$</td></tr>
<tr><td>5th</td><td>$C_3 = 1$
$C_2 = B_4 - (S_1+S_2)$
$C_1 = (B_2 - C_0 - C_2S_1S_2)/(S_1+S_2)$
$C_0 = B_0/S_1S_2$</td></tr>
<tr><td>6th</td><td>$C_4 = 1$
$C_3 = B_5 - (S_1+S_2)$
$C_2 = B_4 - C_3(S_1+S_2) - S_1S_2$
$C_1 = (B_2 - C_0 - C_2S_1S_2)/(S_1+S_2)$
$C_0 = B_0/S_1S_2$</td></tr>
<tr><td>nth</td><td>$C_{n-2} = 1$
$C_{n-3} = B_{n-1} - C_{n-2}(S_1+S_2)$
$C_{n-4} = B_{n-2} - C_{n-3}(S_1+S_2) - C_{n-2}S_1S_2$
$\vdots \quad \vdots \quad \vdots \quad \vdots$
$C_2 = B_4 - C_3(S_1+S_2) - C_4S_1S_2$
$C_1 = [B_2 - C_0 - C_2S_1S_2]/(S_1+S_2)$
$C_0 = B_0/S_1S_2$</td></tr>
</table>

Table 1-3 Functions of the variable coefficients f_1 and f_3. The functions contained in this table are assumed only B_1 and B_3 are adjustable. $(-S_1)$ and $(-S_2)$ are the arbitrarily chosen roots and C's are coefficients of the reduced characteristic equation.

order of the equation	coeff. of the reduced chara. equation	Functions of the variable coeff.
3rd	$C_1 = 1$ $C_0 = B_2 - (S_1 + S_2)$	
4th	$C_0 = B_2 - C_1(S_1 + S_2) - C_2 S_1 S_2$ $C_1 = B_3 - C_2(S_1 + S_2)$ $C_2 = 1$	
5th	$C_0 = B_2 - (S_1 + S_2)C_1 - S_1 S_2 C_2$ $C_1 = B_3 - (S_1 + S_2)C_2 - S_1 S_2 C_3$ $C_2 = B_4 - (S_1 + S_2)$ $C_3 = 1$	$f_0 = C_0 S_1 S_2$ $f_1 = C_0(S_1 + S_2) + C_1 S_1 S_2$
nth	$C_0 = B_2 - (S_1 + S_2)C_1 - S_1 S_2 C_2$ $C_1 = B_3 - (S_1 + S_2)C_2 - S_1 S_2 C_3$ $\vdots$ $C_i = B_{i+2} - (S_1 + S_2)C_{i+1} - S_1 S_2 C_{i+2}$ $\vdots$ $C_{n-3} = B_{n-1} - (S_1 + S_2)$ $C_{n-2} = 1$	

Table 1-4. Functions of the variable coefficients f_o and f_1. The functions in this table are assumed B_o and B_1 are adjustable simultaneously. Where $(-S_1)$ and $(-S_2)$ are the arbitrarily chosen roots and C's are the coefficients of the reduced characteristic equation.

order of the equation	coeff. of the reduced chara. equation	functions of the variable coeff.
5 th	$C_2 = 1$ $C_1 = B_4 - (S_1 + S_2 + S_3)$ $C_0 = B_0 / S_1 S_2 S_3$	$f_1 = C_0(S_1S_2 + S_2S_3 + S_3S_1) + C_1 S_1S_2S_3$ $f_2 = C_0(S_1 + S_2 + S_3) + C_1(S_1S_2 + S_2S_3 + S_3S_1) + C_2 S_1S_2S_3$ $f_3 = C_0 + C_1(S_1 + S_2 + S_3) + C_2(S_1S_2 + S_2S_3 + S_3S_1) + C_3 S_1S_2S_3$
6 th	$C_3 = 1$ $C_2 = B_5 - (S_1 + S_2 + S_3)$ $C_1 = B_4 - C_2(S_1 + S_2 + S_3) - C_3(S_1S_2 + S_2S_3 + S_3S_1)$ $C_0 = B_0$ $C_0 = B_0 / S_1 S_2 S_3$	
n th	$C_{n-3} = 1$ $C_{n-4} = B_{n-1} - C_{n-3}(S_1 + S_2 + S_3)$ $C_{n-5} = B_{n-2} - C_{n-4}(S_1 + S_2 + S_3) - C_{n-3}(S_1S_2 + S_2S_3 + S_3S_1)$ $C_{n-6} = B_{n-3} - C_{n-5}(S_1 + S_2 + S_3) - C_{n-4}(S_1S_2 + S_2S_3 + S_3S_1)$ $\vdots \quad \vdots \quad \vdots \quad \vdots$ $C_1 = B_4 - C_2(S_1 + S_2 + S_3) - C_3(S_1S_2 + S_2S_3 + S_3S_1)$ $C_0 = B_0 / S_1 S_2 S_3$	

Table 1-5. Functions of the variable coefficients f_1, f_2 and f_3. The functions contained in this table are assumed B_1, B_2 and B_3 are adjustable simultaneously. Where $(-S_1)$, $(-S_2)$ and $(-S_3)$ are the arbitrarily chosen roots and the C's are the coefficients of the reduced characteristic equation.

order of the equation	coeff. of the reduced chara. equation	functions of the variable coeff.
4th	$C_0 = B_3 - (S_1+S_2+S_3)$ $C_1 = 1$	$f_0 = C_0 S_1S_2S_3$ $f_1 = C_0(S_1S_2+S_2S_3+S_3S_1) + C_1 S_1S_2S_3$ $f_2 = C_0(S_1+S_2+S_3) + C_1(S_1S_2+S_2S_3+S_3S_1) + C_2 S_1S_2S_3$
5th	$C_0 = B_3 - C_1(S_1+S_2+S_3) - (S_1S_2+S_2S_3+S_3S_1)$ $C_1 = B_4 - (S_1+S_2+S_3)$ $C_2 = 1$	
n th	$C_0 = B_3 - C_1(S_1+S_2+S_3) - C_2(S_1S_2+S_2S_3+S_3S_1)$ $C_1 = B_4 - C_2(S_1+S_2+S_3) - C_3(S_1S_2+S_2S_3+S_3S_1)$ $C_2 = B_5 - C_3(S_1+S_2+S_3) - C_4(S_1S_2+S_2S_3+S_3S_1)$ $\vdots$ $C_{n-4} = B_{n-1} - (S_1+S_2+S_3)$ $C_{n-3} = 1$	

Table 1-6. Functions of the variable coefficients f_0, f_1 and f_2. The functions contained in this table are assumed B_0, B_1, and B_2 are adjustable simultaneously. Notations are the same as Table 1-5.

CHAPTER II

REDUCED CHARACTERISTIC EQUATION

2-1. Introduction: The reduced characteristic equation determines all the roots of the controlled characteristic equation and consequently determines the response of the system. Usually the zeros of the closed loop system are known, then the total performance of the system depends upon the choice of the poles (roots of the characteristic equation). In order to establish a guide for the choice of the roots, the properties of the reduced characteristic equation are investigated. For the reason of simplicity and convenience most of the examples in this chapter take the coefficients themselves as variables. For the case of the adjustable parameters scattered in all the coefficients, the analyses are the same. In fact, where the analysis of the reduced characteristic equation alone is concerned, it does not make any difference about what type of compensator is used. The analyses in this chapter also assumed that no restrictions are imposed in the variables, consequently no restrictions are imposed on the arbitrary roots.

2-2. Coefficients as variables.

Treating some of the coefficients of the controlled characteristic equation as variables provides a convenient and relatively systematic means for the purposes of analyses. However, the actual applications have some difficulties. Theoretically, the variables can be transformed analytically or graphically from one set of variables to another by the root-coefficient relation and the definition of variables. Thus the variables which originally are coefficients of a characteristic equation can be transformed to another set of variables such as the actual adjustable parameters. However, the transformation may be very complicated

because of the inherent nature of the algebraic equations. In actual design, it is unnecessary, because the actually adjustable parameters can be treated as variables directly by the partitioning processes described in the last chapter.

Mitrovic's method treats two coefficients as variables. The application of Mitrovic's method to the case in which the actually adjustable parameters enter some of the presumed fixed coefficients requires that transformations or approximations must be made either graphically or analytically. Some of the work to this end has been done.[2,4,5]

2-3. Effectiveness of the adjustable coefficients.

The effectiveness of stabilizing ability of adjustable coefficients are discussed in this section. Since only a qualitative analyses, i.e., the tendency of stabilization is required, the root-locus method is applied to the reduced characteristic equation.

Consider a fourth order characteristic equation (2-1) the effectiveness

$$S^4 + B_3 S^3 + B_2 S^2 + B_1 S + B_0 = 0 \tag{2-1}$$

of adjustable coefficients are analyzed in several cases:

Case (1): B_2 adjustable. Since only B_2 is adjustable, then the roots are constrained by B_3, B_1 and B_0. Assume the reduced characteristic equation is

$$S^3 + C_2 S^2 + C_1 S + C_0 = 0 \tag{2-2}$$

and the arbitrarily chosen root is $-S_1$, then

$$B_3 = C_2 + S_1$$

$$B_1 = C_0 + C_1 S_1$$

$$B_0 = C_0 S_1$$

From the above equations, solve for C_1, C_2 and C_o, obtain

$$C_2 = B_3 - S_1 \tag{2-3-1}$$

$$C_1 = \frac{1}{S_1}(B_1 - C_o) = \frac{B_1}{S_1} - \frac{B_o}{S_1^2} \tag{2-3-2}$$

(2-3-3)

Substitute (2-3) into equation (2-2), one obtains

$$S^3 + (B_3 - S_1)S^2 + \left(\frac{B_1}{S_1}\right)S - \frac{B_o}{S_1^2}(S - S_1) = 0 \tag{2-4}$$

Equation (2-4) is the reduced characteristic equation with the coefficients expressed in terms of the arbitrary root $-S_1$. Assume S_1 has been chosen, the tendency of the root locus of equation (2-4) by varying B_o is to be investigated. Equation (2-4) in root-locus form is

$$\frac{\frac{B_o}{S_1^2}(S - S_1)}{S\left[S^2 + (B_3 - S_1)S + \frac{B_1}{S_1}\right]} = 1 \tag{2-4}$$

Consider equation (2-4) and assume $S_1 \neq 0$, then there is a zero in the right-half plane and the angle-criteria of the root-locus is $2\pi n$ (n = 0, 1, 2,) instead of $m\pi$ (m = 1, 3, 5...). The general root-locus pattern is shown in Fig. 2-1, in which R_1 and R_2 are the roots of the equation

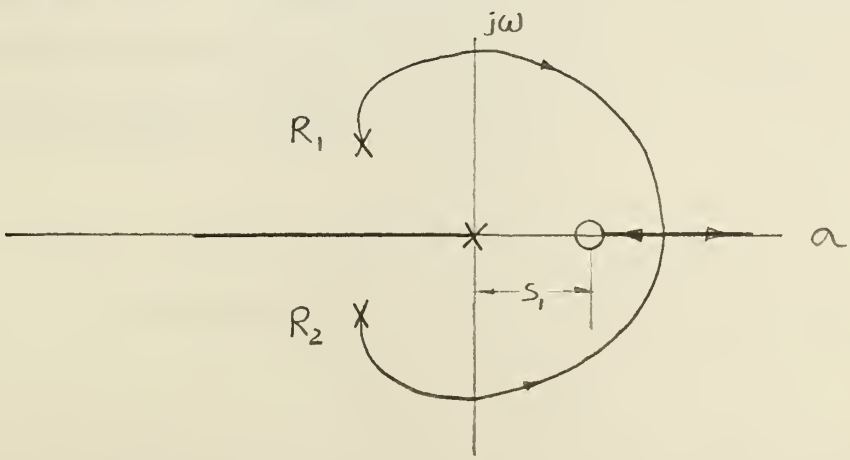


Fig. 2-1 Root-locus of equation (2-4). Case of B_2 adjustable.

Case (2): B_3 adjustable. Now consider the same fourth order characteristic equation (2-1) in which only the coefficient of the 3rd derivative term is adjustable. By the same manipulation as Case (1), the coefficients of the reduced characteristic equations are obtained as follows:

$$C_1 = B_1/S_1 - B_0/S_1^2 \tag{2-5-1}$$

$$C_2 = B_2/S_1 - B_1/S_1^2 + B_0/S_1^3 \tag{2-5-2}$$

$$C_0 = B_0/S_1 \tag{2-5-3}$$

Substitute (2-5) into (2-2), the reduced characteristic equation becomes

$$S^3 + \left(\frac{B_2}{S_1} - \frac{B_1}{S_1^2}\right)S^2 + \left(\frac{B_1}{S_1}\right)S + \frac{B_0}{S_1^3}\left(S^2 - S_1 S + S_1^2\right) = 0 \tag{2-6}$$

In root-locus form for B_o as variable, (2-6) becomes

$$\frac{\frac{B_0}{S_1^3}\left(S^2 - S_1 S + S_1^2\right)}{S\left[S^2 + \left(\frac{B_2}{S_1} - \frac{B_1}{S_1^2}\right)S + \frac{B_1}{S_1}\right]} = -1 \tag{2-7}$$

Assume $S_1 \neq 0$, equation (2-7), has two complex zeros in the right half plane and the angle-criteria of the root-locus is $\pm$ 180°. The root locus pattern is shown in Fig. 2-2.

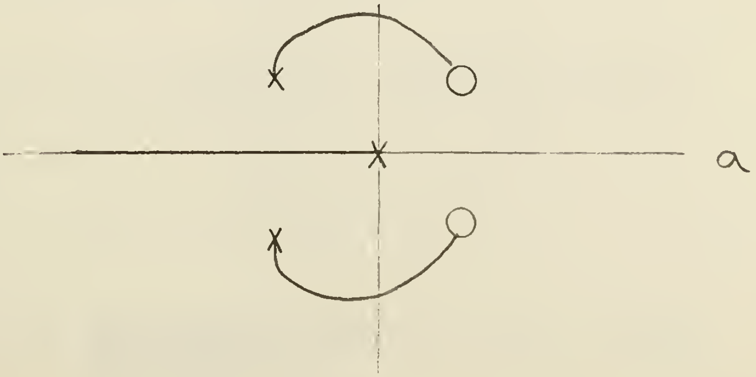


Fig. 2-2. Root-locus of equation (2-7), B_3 adjustable

Case (3): B_1 and B_3 adjustable.

Consider the same fourth order equation but with B_3 and B_1 adjustable. Since there are two coefficients adjustable, the reduced characteristic equation is of order 2. Follow the same derivation, the coefficients of the reduced characteristic equation are:

$$C_1 = \frac{1}{S_1 + S_2}\left(B_2 - S_1 S_2 - C_0 \right) \tag{2-8-1}$$

$$C_0 = \frac{B_0}{S_1 S_2} \tag{2-8-2}$$

where $-S_1$ and $-S_2$ are arbitrarily chosen roots and $S_1 + S_2 \neq 0$, $S_1 S_2 \neq 0$ are assumed. The reduced characteristic equation is

$$S^2 + \frac{1}{S_1 + S_2}\left(B_2 - S_1 S_2 - \frac{B_0}{S_1 S_2} \right) S + \frac{B_0}{S_1 S_2} = 0 \tag{2-9}$$

In root-locus form for B_0 as variable is

$$\frac{\frac{B_0}{S_1 S_2 (S_1 + S_2)} \left[S - (S_1 + S_2) \right]}{S \left[S + \frac{B_2 - S_1 S_2}{S_1 + S_2} \right]} = 1 \tag{2-10}$$

This equation again has one zero in the right half plane and the angle-criteria of the root-locus is $2\pi n$ (n = 0, 1, 2,...). The general root-locus pattern is shown in Fig. 2-3.

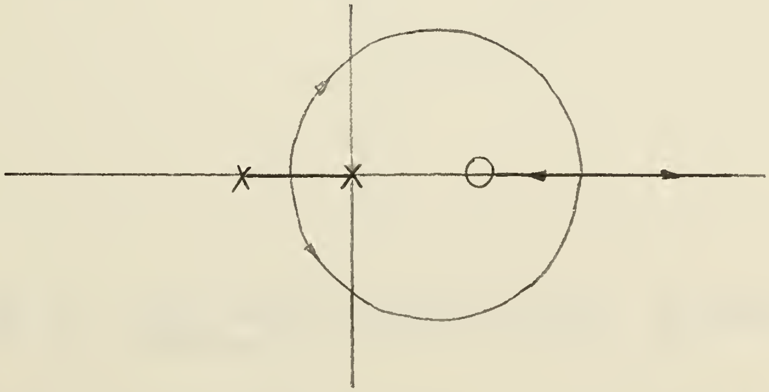


Fig. 2-3. Root-locus of equation (2-10). B_1 and B_3 adjustable.

Case (4): B_2 and B_3 adjustable.

In this case, the reduced characteristic equation is of order 2 and the coefficients from Table 1-2 are as follows:

$$C_1 = \frac{B_1}{S_1 S_2} - \frac{S_1 + S_2}{(S_1 S_2)^2} B_0 \tag{2-11-1}$$

$$C_0 = \frac{B_0}{S_1 S_2} \tag{2-11-2}$$

In root-locus form, the reduced characteristic equation becomes

$$\frac{\frac{(S_1 + S_2)}{(S_1 S_2)^2} B_0 \left(S - \frac{S_1 S_2}{S_1 + S_2}\right)}{S\left(S + \frac{B_1}{S_1 S_2}\right)} = 1 \tag{2-12}$$

Equation (2-12) has one zero in the right half plane and the angle-criteria of the root-locus is $2\pi n$ $(n = 0, 1, 2, \ldots)$. The general root-locus pattern is shown in Fig. 2-4 when $S_3 + S_4 \neq 0$.

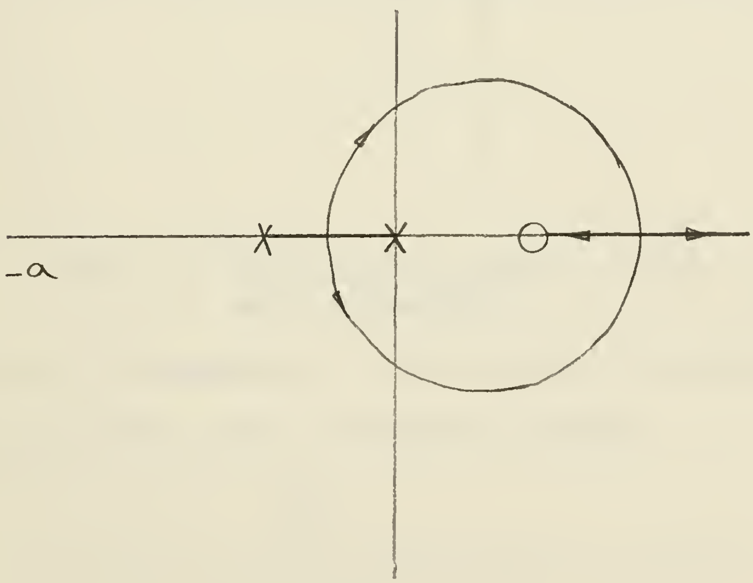


Fig. 2-4. Root-locus of equation (2-12). B_2 and B_3 adjustable.

Case (5): B_1 and B_2 adjustable - From Table 1-1. The coefficient of the reduced characteristic equations are as follows:

$$C_1 = B_3 - (S_1 + S_2) \tag{2-13-1}$$

$$C_0 = \frac{B_0}{S_1 S_2} \tag{2-13-2}$$

The reduced characteristic equation in root-locus form is

$$\frac{\frac{1}{S_1 S_2} B_0}{S\left[S + (B_3 - S_1 - S_2)\right]} = -1 \tag{2-14}$$

The general root-locus is shown in Fig. 2-5.

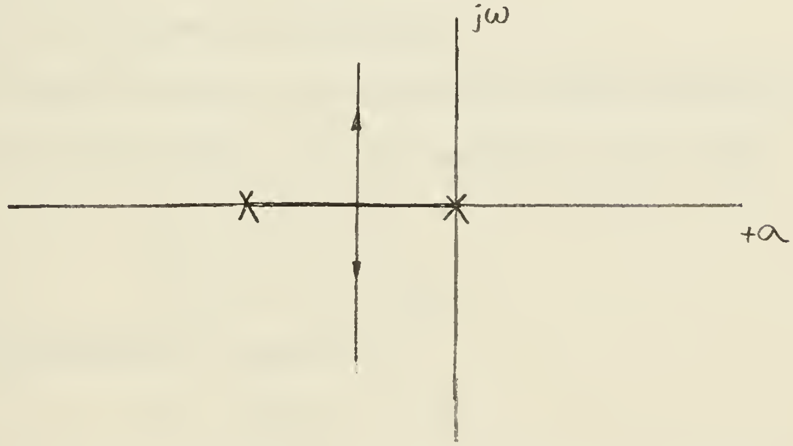


Fig. 2-5 Root-locus of equation (2-14). B_1 and B_2 adjustable.

Case (6): B_1 adjustable. In the same way, the reduced characteristic equation in root-locus form is obtained as follows:

$$\frac{\frac{B_0}{S_1}}{S\left[S^2 + (B_3 - S_1)S + (B_2 + S_1^2 - B_3 S_1)\right]} = -1 \tag{2-15}$$

The general root-locus pattern is shown in Fig. 2-6.

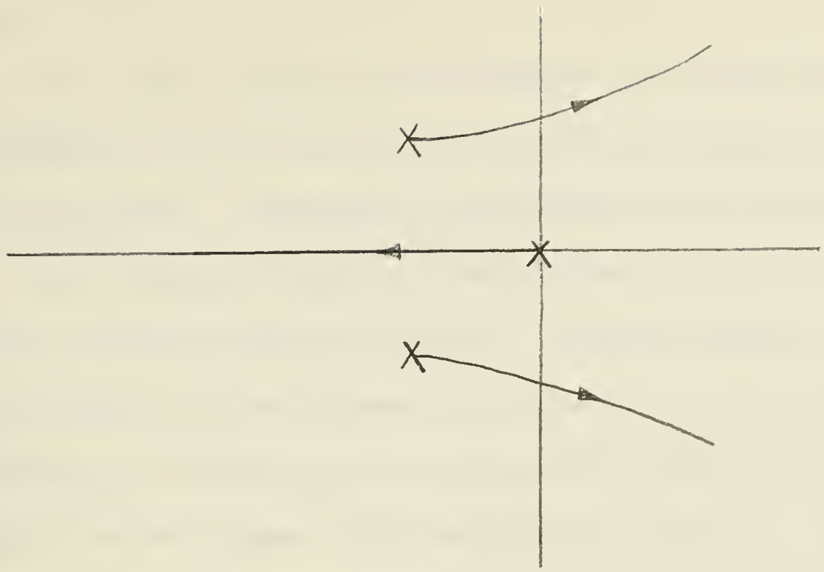


Fig. 2-6. Root-locus of equation (2-15) B_1 adjustable.

Case (7): B_1, B_2 and B_3 adjustable:

In this case three roots are arbitrary, only one root is constrained. Assume S_4 is the constrained root, and define $C_o = S_4$, then

$$B_o = S_1 S_2 S_3 C_o$$

$$C_o = B_o / S_1 S_2 S_3$$

The reduced characteristic equation is

$$S + \frac{B_o}{S_1 S_2 S_3} = 0$$

If no restrictions are imposed on the adjustable coefficients, then for any choice of S_1, S_2 and S_3 ($S_1 \neq 0$, $S_2 \neq 0$, $S_3 \neq 0$), the system can be stabilized for any value of B_o.

From the above analyses and the general pattern of the root-locus three conclusions can be drawn:

(1) If the adjustable coefficients are not in the lowest order or not in sequence, as the cases (1), (2), (3) and (4), right half plane zeros are introduced in the root-locus plot. Consequently, the stabilizing ability is less than that if no right half-plane zeros would be

introduced.

(2) If the adjustable coefficients are in the lowest order and in sequence, as the cases (5), (6) and (7), no right half-plane zeros are introduced. Consequently the stabilization is more effective.

(3) The most effective scheme is the case where all of the coefficients are adjustable as case (7). The most ineffective scheme is the case where only the highest coefficient is adjustable as case (1).

In general, if there are "r" unadjustable coefficients which have an order lower than the highest order adjustable coefficient, then the reduced characteristic equation in root-locus plot for B_o as variable has "r" zeros in the right half plane. Because of this property, it seems that if the adjustable coefficients of a characteristic equation are not the lower order and not in sequence, the stabilizing tendency of these sets is less than for the case of adjustable coefficients which are of lower order and in sequence. Consider case (5), in which B_1 and B_2 are adjustable, the reduced characteristic equation is of the second order and there are no zeros in the right half plane, consequently the system can always be stabilized for any value of K. However, for cases (1) to (4), it cannot always stabilize the system. In general, for any order system if the reduced characteristic equation is of second order and all adjustable coefficients are of lower order and in sequences, then the system can be stabilized for any forward gain K. However, this is not the case if the adjustable coefficients are not of lower order and in sequence. Moreover, the more zeros introduced in the right half plane in the process of derivation of the reduced characteristic equation, the less the tendency to stabilize the system. This implies that the more the unadjustable coefficients which are of lower order than the highest order adjustable

coefficient, the less the effectiveness of the control parameters. This is the inherent property of algebraic equations. It is because of this property that many approximations can be made by ignoring those control parameters which affect only the coefficients of higher derivatives. If the design of a control system is based on the characteristic equation, those coefficients of lower order derivatives should be considered first if it is possible.

This analyses also explains the well known fact that a cascade filter is more effective than lower order derivative feedback, especially for high order systems. For a cascade filter, the filter parameters are scattered in all coefficients of the characteristic equation in the nature of algebraic equations, while derivative feedback enters some of the coefficients only. This analyses also explains the fact that tachometer feedback is good for low order systems, but may even hurt the performance of high order systems.

2-4. Root-Region of Stability.

The reduced characteristic equation is a function of the arbitrary roots. Mathematically for any choice of the arbitrary roots, there exists solution of the equation. However, in control systems, the roots of the characteristic equations are required to be in the left half plane, this requirement therefore confines the arbitrary roots in a certain region in the left-half plane. This region here is called "region of stability" and may serve as a guide for the choice of the arbitrary roots.

If the number of variables is one, this region is a line or lines in the S-plane (root-locus). When the number of variables is more than one, this region is an area or several areas on S-plane. The boundary of this region can be calculated by Routh's criterion subject to the reduced

characteristic equation. The pattern of 4th and 5th order equations for $f_1 - f_2$, $f_2 - f_3$ and $f_1 - f_3$ as variables are shown in Fig. 2-7 and Fig. 2-8.

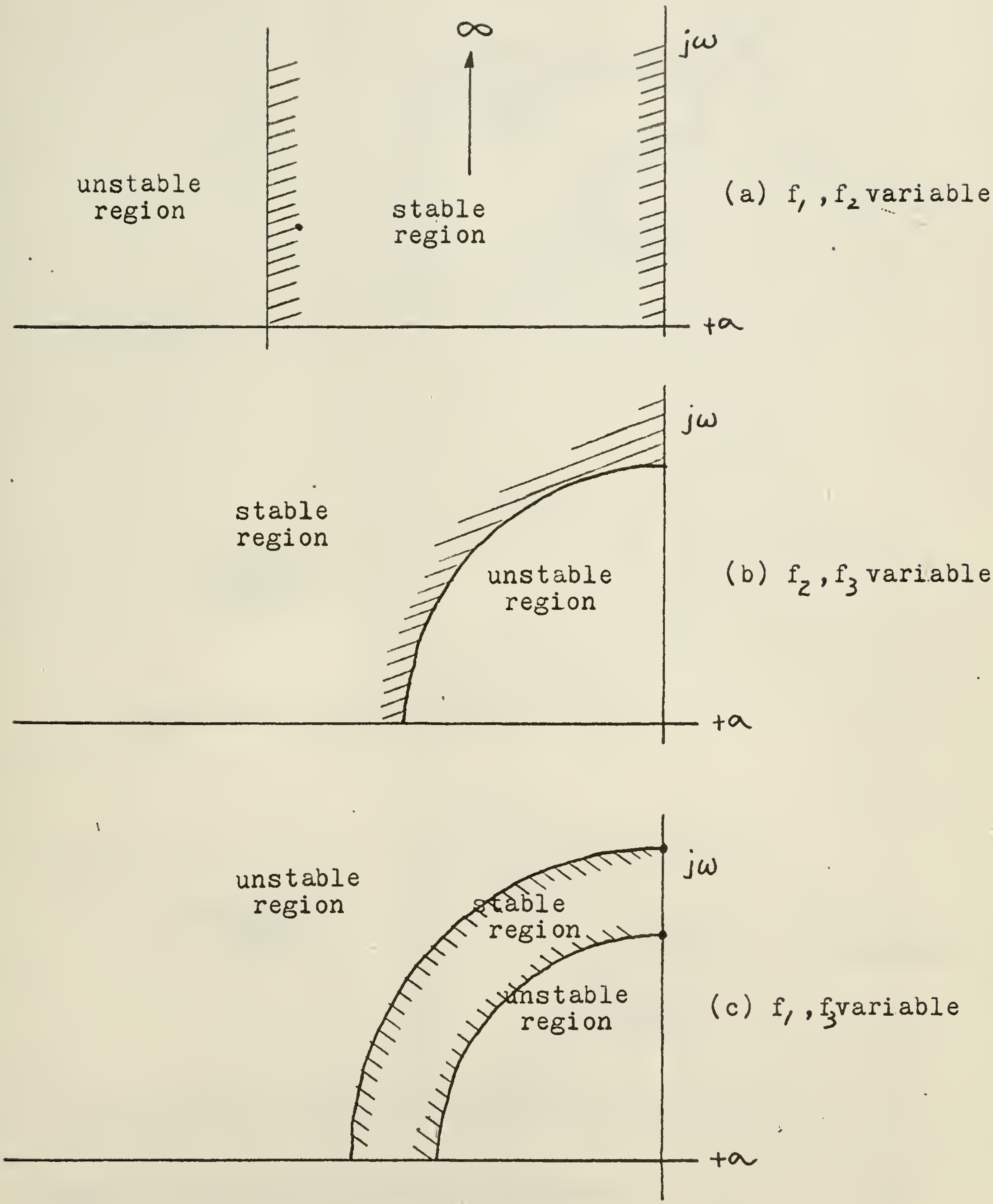


Fig. 2-7 Root region of a 4th order characteristic equation

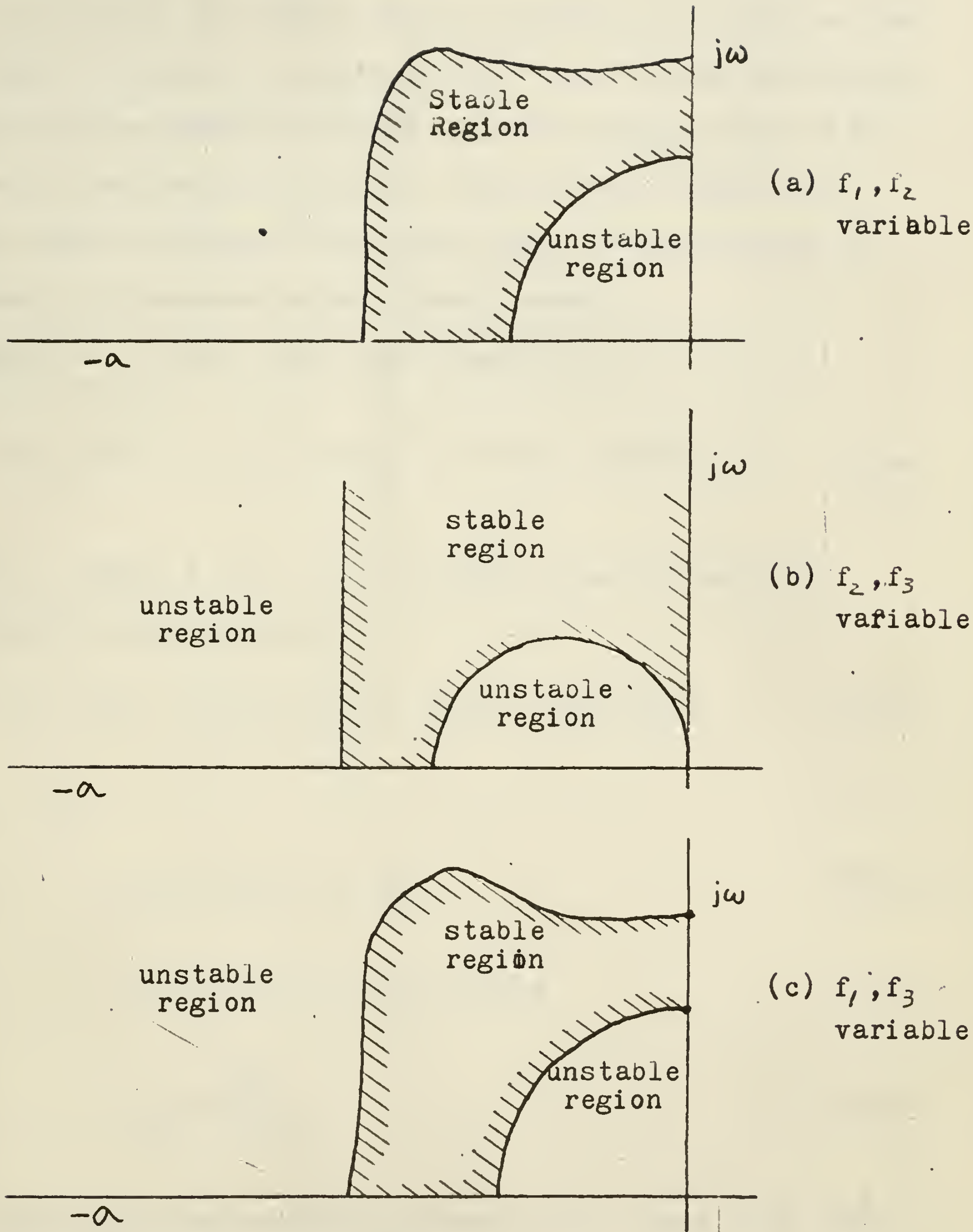


Fig. 2-8 Root region of a 5th order equation

In Fig. 2-7 and Fig. 2-8, if a pair of complex roots are in the stable region, then all the roots are in that region. If a pair of complex roots are in the unstable region, then there is at least one root in the right half plane. If a pair of complex roots are on the stability boundary line, then there are at least one pair of complex roots on the $j\omega$ axis or one root at the origin. The arbitrarily chosen roots, therefore, must be confined in the stable region. The calculation of the boundary is illustrated in the following example.

Example 2-1: Given a fifth order equation

$$S^5 + 16S^4 + 128S^3 + 520S^2 + 1300S + 2000 = 0 \qquad (2\text{-}16)$$

Case 1: Assume f_2 and f_3 are variables, then the controlled characteristic equation becomes

$$S^5 + 16S^4 + f_3 S^3 + f_2 S^2 + 1300S + 2000 = 0 \qquad (2\text{-}17)$$

From Table 1-2:

$$C_2 = B_4 - (S_1 + S_2) \qquad (2\text{-}18\text{-}1)$$

$$C_1 = \frac{1}{S_1 S_2}\left[B_1 - C_0 (S_1 + S_2)\right] \qquad (2\text{-}18\text{-}2)$$

$$C_0 = \frac{B_0}{S_1 S_2} \qquad (2\text{-}18\text{-}3)$$

where $-S_1$ and $-S_2$ are the arbitrarily chosen roots. Assume these roots are confined on a constant ζ line, then

$$S_1 + S_2 = 2\zeta\omega_n \quad , \qquad S_1 S_2 = \omega_n^2$$

Substitute into (2-18)

$$C_2 = B_4 - 2\zeta\omega_n \tag{2-19-1}$$

$$C_1 = \frac{1}{\omega_n^2}\left[B_1 - \frac{2\zeta B_0}{\omega_n}\right] \tag{2-19-2}$$

$$C_0 = \frac{B_0}{\omega_n^2} \tag{2-19-3}$$

where $B_4 = 16$, $B_o = 2000$, $B_1 = 1300$ from the given equation (2-17).

The Routh's criterion for the 3rd order reduced characteristic equation is

$$C_1 C_2 > C_0 \tag{2-20}$$

Substitute (2-19) and the values of B_1, B_4 and B_o into (2-20), and manipulate, the Routh's criterion in equality becomes

$$2.6\zeta\omega_n + \frac{64\zeta}{\omega_n} < 18.8 + 8\zeta^2 \tag{2-21}$$

Let the left side of the (2-12) be f, then

$$f = 2.6\zeta\omega_n + \frac{64\zeta}{\omega_n}$$

The f versus ω_n plot for given value of ζ is shown in Fig. 2-9.

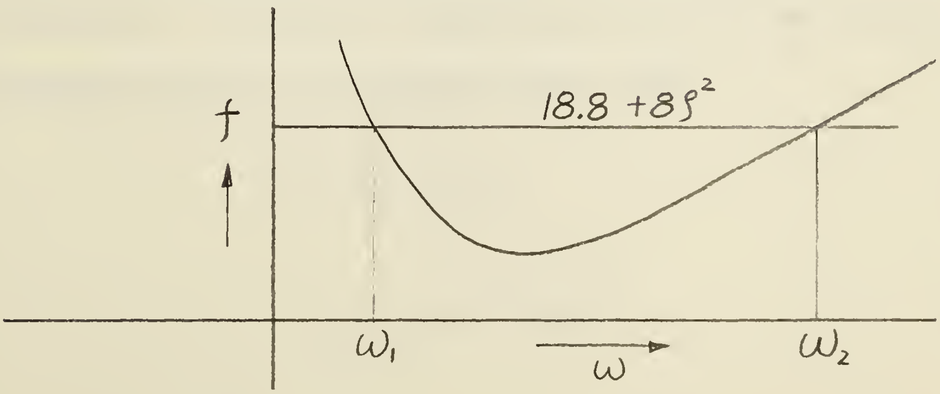


Fig. 2-9 Graphical solution of equation (2-21)

From the inequality (2-21), if $\omega_1 \leq \omega_n \leq \omega_2$, then all the roots of the reduced characteristic equation are on the left-half plane. ω_1 and ω_2 define the stability boundary of the arbitrarily chosen roots and are the solution of the following equation

$$2.6\,\zeta\omega_n + \frac{64\zeta}{\omega_n} = 18.8 + 8\zeta^2 \qquad (2\text{-}22)$$

Solutions of (2-22) for $0 < \zeta < 1$ are tabulated in Table 2-1, and the boundary for root region are plotted in Fig. 2-10.

Table 2-1. Solutions of equation (2-22)

ζ	ω_1	ω_2
1	3.7	6.62
0.9	3.28	7.53
0.7	2.4	10.5
0.6	2.05	12.9
0.5	1.7	14.3
0.4	1.35	17.9
0.3	1.05	23.95
0.2	0.65	36.1
0.1	0.35	72

Case 2: f_1, f_2 variable

Since the coefficients of the reduced characteristic equation are expressed by the arbitrarily chosen roots, the mapping contour can be chosen arbitrarily. In order to demonstrate this, let the two arbitrarily chosen roots be on a constant α line, then

$$S_1 + S_2 = 2\alpha \qquad (2\text{-}23\text{-}1)$$

$$S_1 S_2 = \alpha^2 + \omega^2 \qquad (2\text{-}23\text{-}2)$$

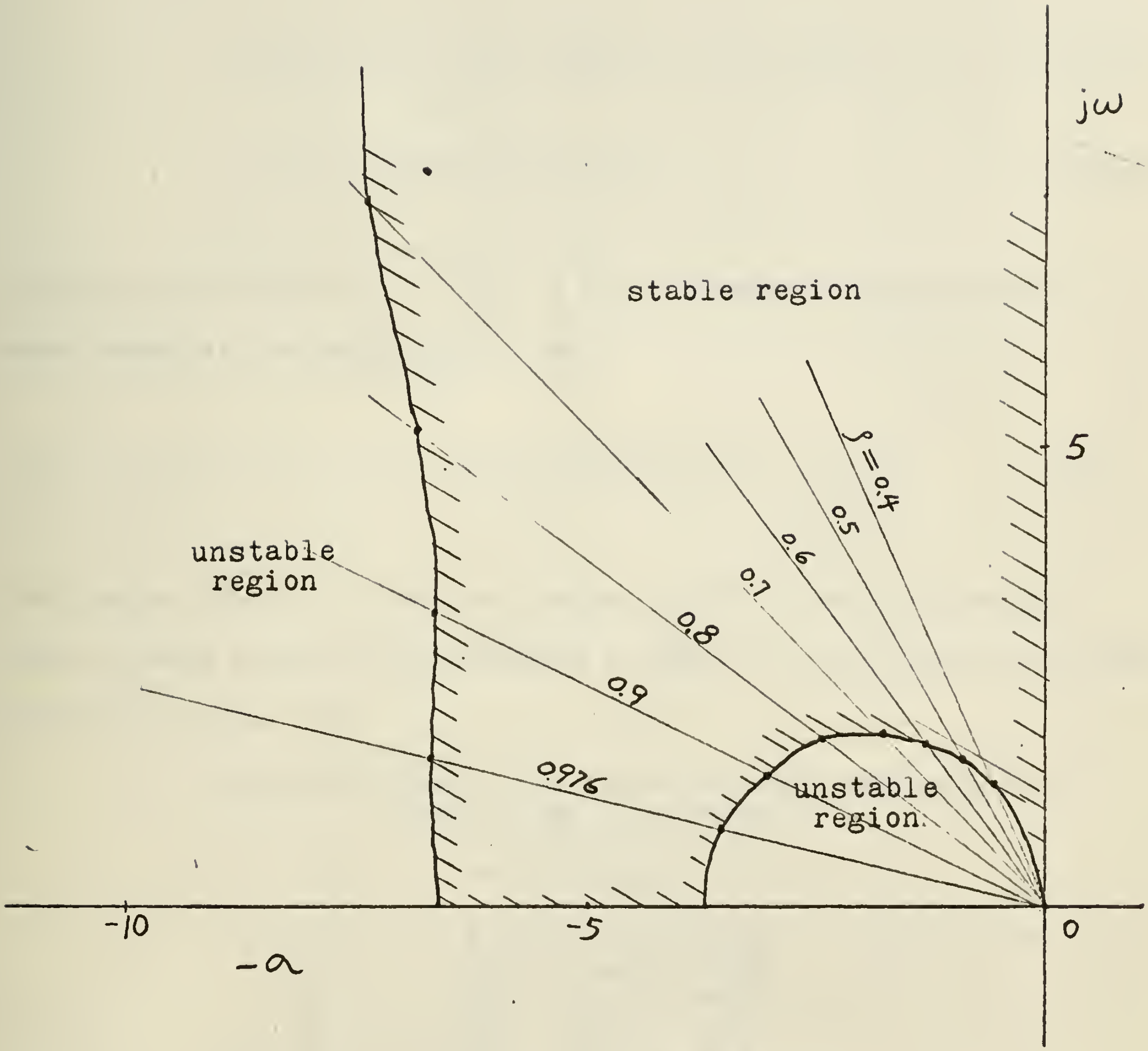


Fig. 2-10 Stable root region of a 5th order equation with variable of example 2-1.

Substitute (2-23) into the coefficients of the reduced characteristic equation in Table 1-2, one obtains

$$C_2 = 16 - 2\alpha \tag{2-24-1}$$

$$C_1 = 520 - (16 - 2\alpha)2\alpha - (\alpha^2 + \omega^2) \tag{2-24-2}$$

$$C_0 = 2000/(\alpha^2 + \omega^2) \tag{2-24-3}$$

The Routh's criterion is $C_2 C_1 > C_0$ Substitute (2-24) into the above inequality and manipulate, obtain

$$(16 - 2\alpha)\left[128 - (16 - 2\alpha)2\alpha - (\alpha^2 + \omega^2)\right](\alpha^2 + \omega^2) > 2000 \tag{2-25}$$

The boundary value of ω for the inequality (2-25) are solved by the same procedure as Case 1 and tabulated in Table 2-2. The stable root region is plotted in Fig. 2-11.

Table 2-2 Stable root region of a 5th order equation with f_1 and f_2 variable.

α	Stable region of ω	
0	$0.975 < \omega < 11.29$	$-0.975 > \omega > -11.29$
2	$-8.57 < \omega < +8.57$	
4	$-6.62 < \omega < +6.62$	
6	$-5.9 < \omega < +5.9$	
7	$-6.3 < \omega < +6.3$	
7.5	$-5.9 < \omega < +6.9$	

Case 3: f_1, f_3 variable.

In this case, from Table 1-3, the coefficients of the reduced characteristic equation are:

$$C_2 = B_4 - (S_1 + S_2) \tag{2-26-1}$$

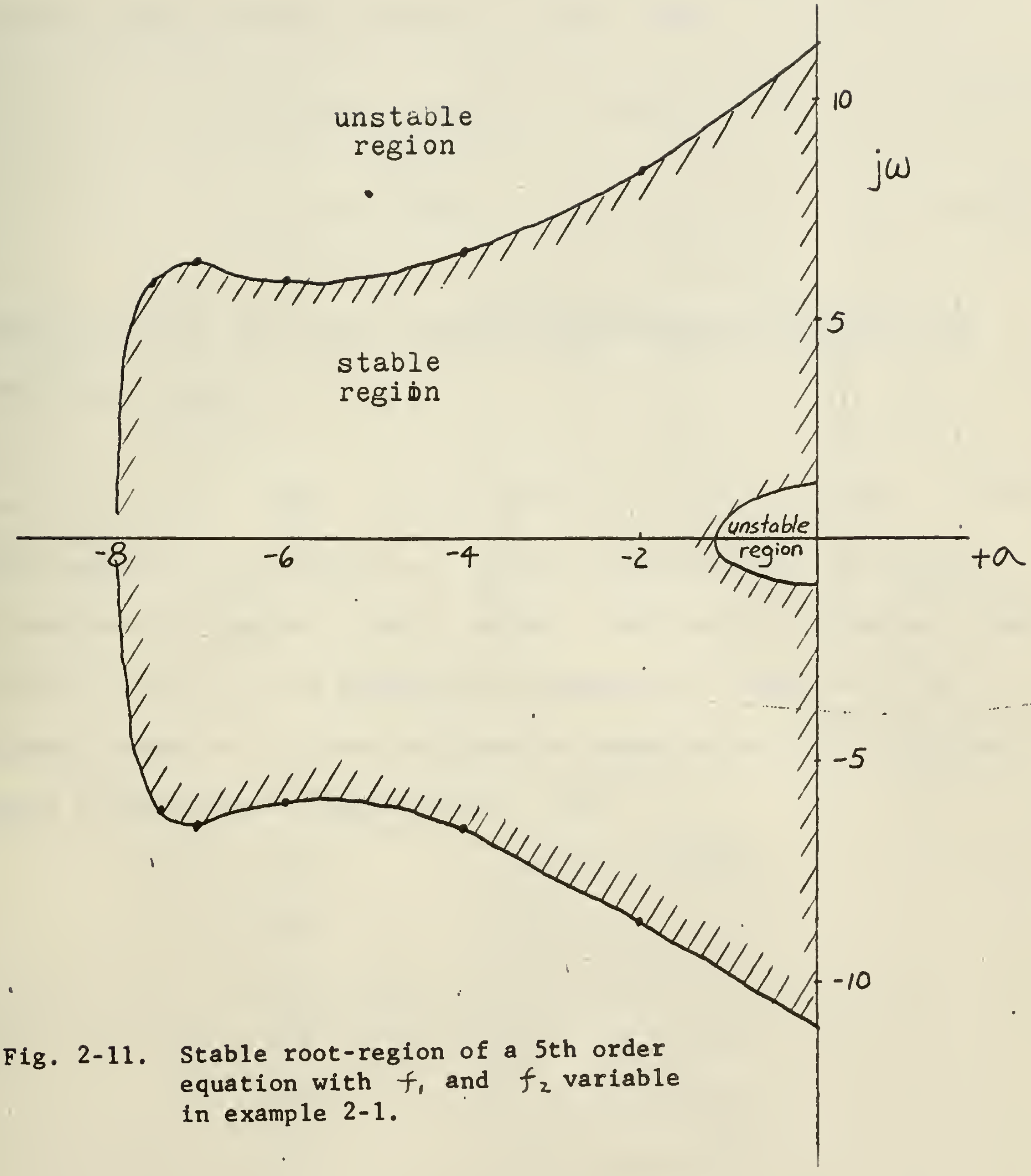


Fig. 2-11. Stable root-region of a 5th order equation with f_1 and f_2 variable in example 2-1.

$$C_1 = \frac{1}{S_1+S_2}\left[B_2 - C_0 - C_2 S_1 S_2\right] \quad , \qquad S_1+S_2 \neq 0 \tag{2-26-2}$$

$$C_0 = B_0 / S_1 S_2 \tag{2-26-3}$$

Assume S_1 and S_2 are on a constant ζ line. Then

$$S_1 + S_2 = 2\zeta\omega_n \ , \quad \zeta \neq 0 \tag{2-27-1}$$

$$S_1 S_2 = \omega_n^2 \tag{2-27-2}$$

Substitute (2-26) and (2-27) into the Routh criterion inequality and manipulate, obtain

$$(16 - 2\zeta\omega_n)\left(520 - \frac{2000}{\omega_n^2} - 16\omega_n^2 + 2\zeta\omega_n^3\right)\omega_n > 2\zeta \times 2000 \tag{2-28}$$

This inequality is a function of ω_n up to the 5th order, the solution is more easily found by a graphic method. The function of the left side of (2-28) for ζ as a parameter are plotted as a function of ω_n, boundary values of ω_n are evaluated as shown in Table 2-3. The root-region of stability is plotted in Fig. 2-12.

ζ	stable region
0.1	$2.2 < \omega_n < 5.45$
0.3	$2.3 < \omega_n < 6$
0.5	$2.4 < \omega_n < 7.1$
0.7	$2.5 < \omega_n < 10.7$
0.9	$2.6 < \omega_n < 8.6$
1.0	$2.68 < \omega_n < 7.3$

Table 2-3. Stable region of 5th equation with f_1, f_3 variable.

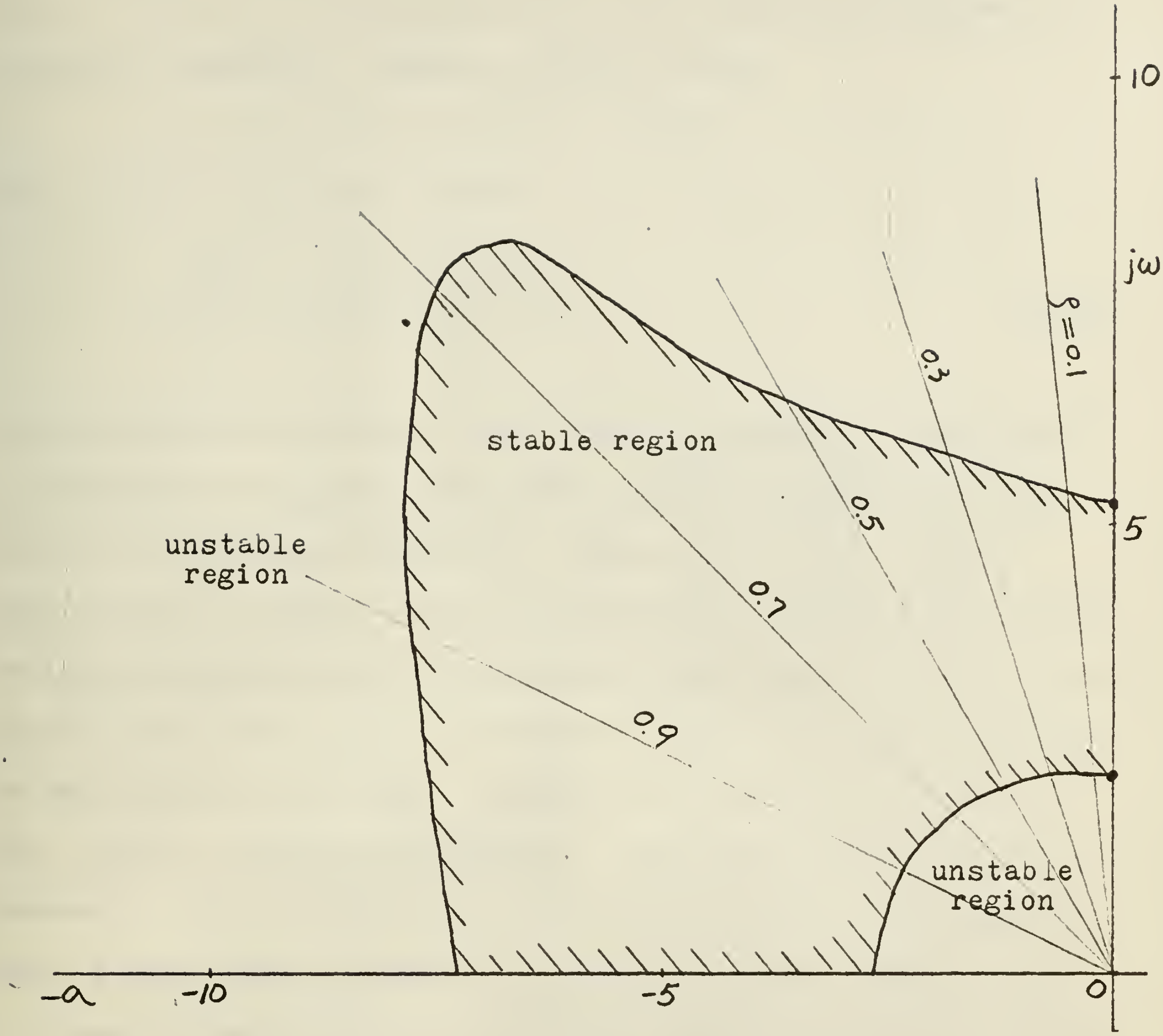


Figure 2-12. Stable root-region of the 5th order equation with f_1, f_3 variable in Example 2-1.

In equation (2-26-2), $S_1 + S_2 \neq 0$ was assumed. This implies $\zeta \neq 0$. For $\zeta = 0$, the coefficient C_1 of the reduced characteristic equation is undefined. Rearrange equation (2-26-2)

$$C_1(S_1+S_2) = B_2 - C_0 - C_1 S_1 S_2$$

For $\zeta = 0$, $S_1 + S_2 = 0$ then

$$B_2 - C_0 - C_2\omega^2 = 0$$

$$B_4\omega^4 - B_2\omega^2 + B_0 = 0 \qquad (2\text{-}28')$$

consider equation (2-28'), the coefficients B_4, B_2 and B_0 are unadjustable coefficients of the given characteristic equation, therefore, the solution of (2-28') defines four points $\pm j\omega_1$, and $\pm j\omega_2$ on the axis instead of a region. Those four points are convergent points for the stability boundary, i.e., if there are any roots on the $j\omega$ axis for this type of control, the roots must be either $\pm j\omega_1$, or $\pm j\omega_2$ regardless of where the other roots are. Consider Fig. 2-13 which is the same as Fig. 2-12, but repeated for convenience. By the definition of stability boundary, if there are any roots on the boundary, there are at least one pair of roots on the $j\omega$ axis or a root at the origin. Assume a point P (inside the stable region or the unstable region) is a root of the equation, as P approaches the boundary in any manner whatever, equation (2-28') implies there are either a pair of complex roots which approach the points $\pm j\omega_1$ or $\pm j\omega_2$, or a root which approaches the origin. In case one and case two of this example, the roots on the $j\omega$ axis have an interval, and the location of the roots in this interval depends upon where the other roots are. In this case, the roots on the $j\omega$ axis are independent of the other roots. In general, if the variable coefficients are not in sequence, then the roots on the $j\omega$ axis have this property. In short, for this

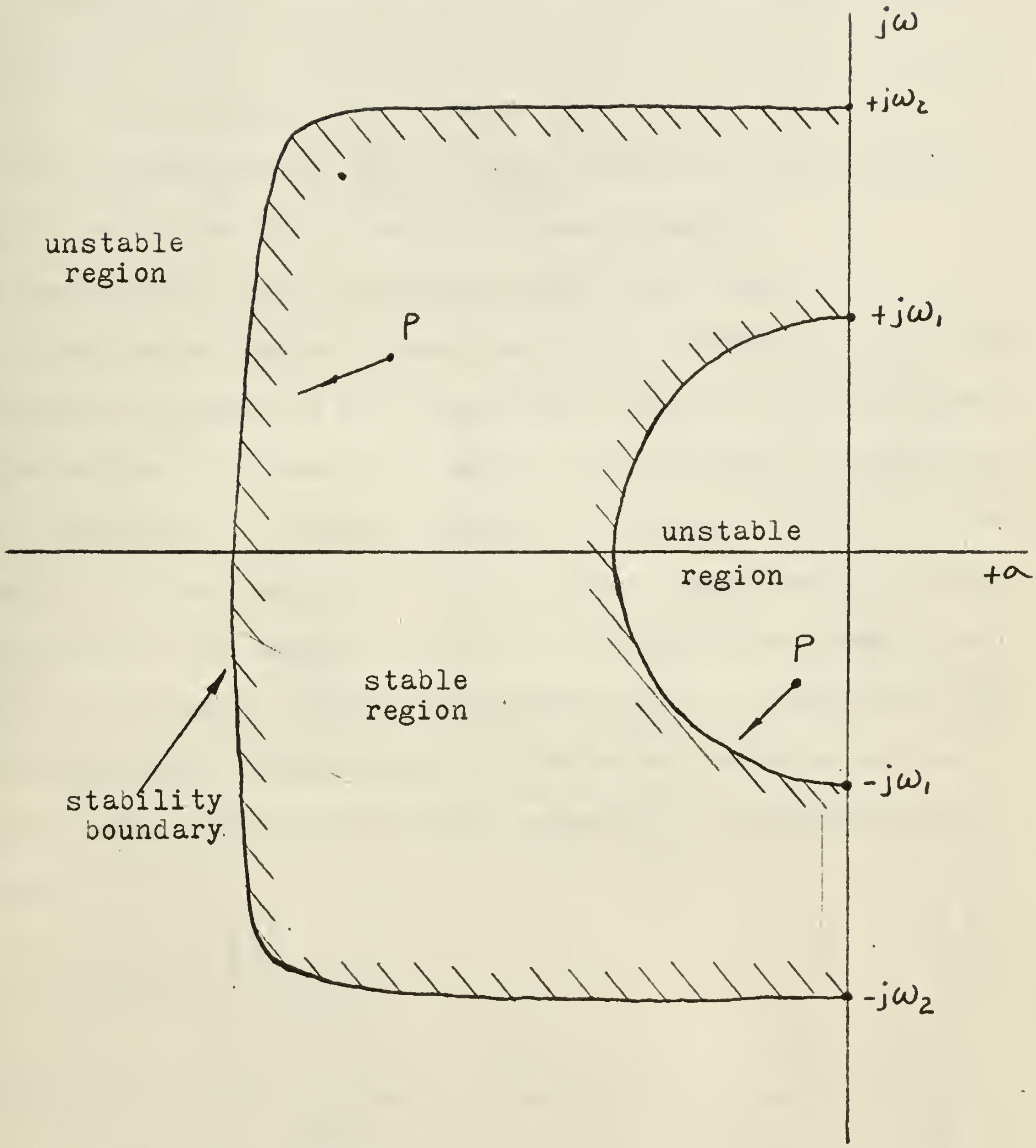


Fig. 2-13 Convergence on the $j\omega$ axis

type of control if the variable coefficients are chosen such that there are roots on the $j\omega$ axis, then those roots on the $j\omega$ axis are completely determined by the fixed coefficients of the given equation.

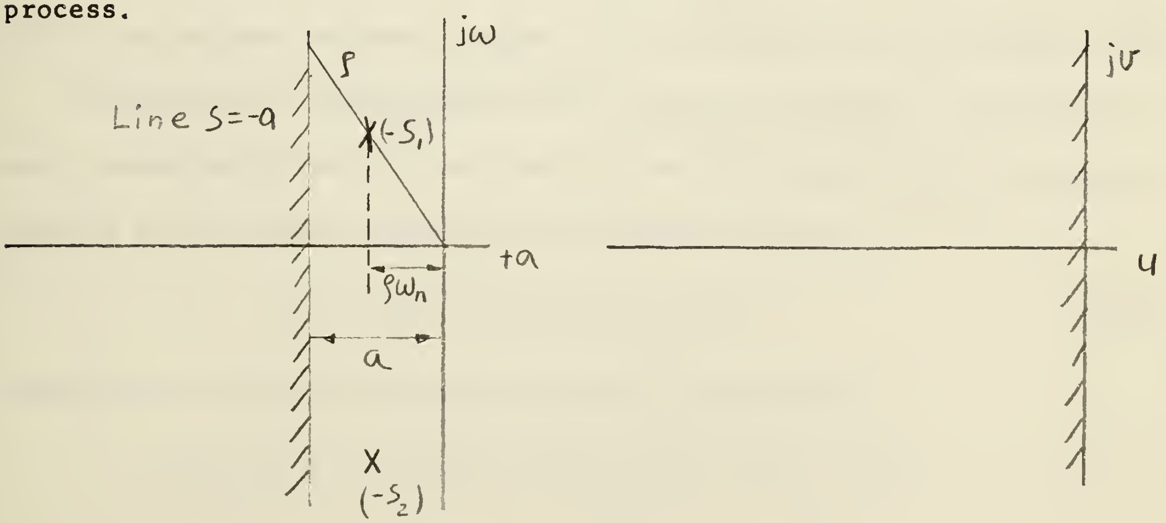
In this example, substitute B_o, B_1 and B_4 into equation (2-28), one obtains

$$\omega_1 = 2.11 \quad , \quad \omega_2 = 5.2$$

The discussions above are for complex roots. For real roots the analysis can be carried out just by the same procedure.

2-5. Dominant Root Region: No Zeros Close to the Origin.

Since most of design is based upon a pair of dominant complex roots, a procedure of evaluating the "dominant root region" is to be discussed. The definition of dominant roots depends upon the degree of dominance and other requirements. It can be defined as one wants. In case of no zeros close to the origin and if the pair of dominant complex roots are so defined such that the negative real part of all other roots (complex and real) of the original characteristic equation must be greater than that of the presumably dominant roots, then the Routh's criterion can be applied to the reduced characteristic equation by a conformal mapping process.



(a) S-plane (b) p-plane

Fig. 2-14 Conformal mapping of S = p-a

Consider Fig. 2-14(a), assume the pair of arbitrary complex roots $(-S_1)$ and $(-S_2)$ are the dominant roots as defined in the last paragraph and also assume all the other roots of the original characteristic equation must have the real parts which are greater than "a" (a is positive real number and greater than or equal to $\zeta\omega_n$), then all the roots of the reduced characteristic equation must be to the left of the line S = -a in the S-plane. By the linear transformation S = p-a, the region to the left of the line S = -a in the S-plane is transformed to the left-half plane in the P-plane. Therefore the Routh's criterion applied to the p-plane, implies all the roots of the reduced characteristic equation are to the left of the line S = -a in the S-plane. In addition, to put all other arbitrarily chosen roots (there may be more than two arbitrarily chosen roots) also to the left of the line S = -a, then all the roots of the original characteristic equation except a pair of complex dominant roots are to the left of the line S = -a. Here the line S = -a was assumed to be fixed if the dominant roots are fixed. In order to find the dominant root-region which satisfy the definition, 'a' must be a variable as the dominant roots are varying. Therefore the application of Routh's criterion to the p-plane defines a region in the S-plane if such a region exists.

To obtain the transformation from S-plane to p-plane, the process is just a substitution of S = p-a into the reduced characteristic equation. Consider a third order reduced characteristic equation

$$S^3 + C_2 S^2 + C_1 S + C_0 = 0 \qquad (2\text{-}29)$$

substitute S = p-a into the above equation, one obtains

$$(p-a)^3 + C_2(p-a)^2 + C_1(p-a) + C_0 = 0$$

manipulating, it becomes

$$p^3 + (C_2 - 3a)p^2 + (C_1 - 2aC_2 + 3a^2)p + (C_0 - C_1 a + C_2 a^2 - a^3) = 0 \tag{2-30}$$

Equation (2-30) is the transformed reduced characteristic equation in the p-plane. Now define:

$$D_2 \triangleq C_2 - 3a \tag{2-31-1}$$

$$D_1 \triangleq C_1 - 2aC_2 + 3a^2 \tag{2-31-2}$$

$$D_0 \triangleq C_0 - C_1 a + C_2 a^2 - a^3 \tag{2-31-3}$$

Where D's are the coefficients of the reduced characteristic equation in p-plane, then equation (2-30) becomes

$$S^3 + D_2 S^2 + D_1 S + D_0 = 0 \tag{2-32}$$

The Routh criterion of equation (2-32) is

$$D_1 D_2 > D_0 \tag{2-33}$$

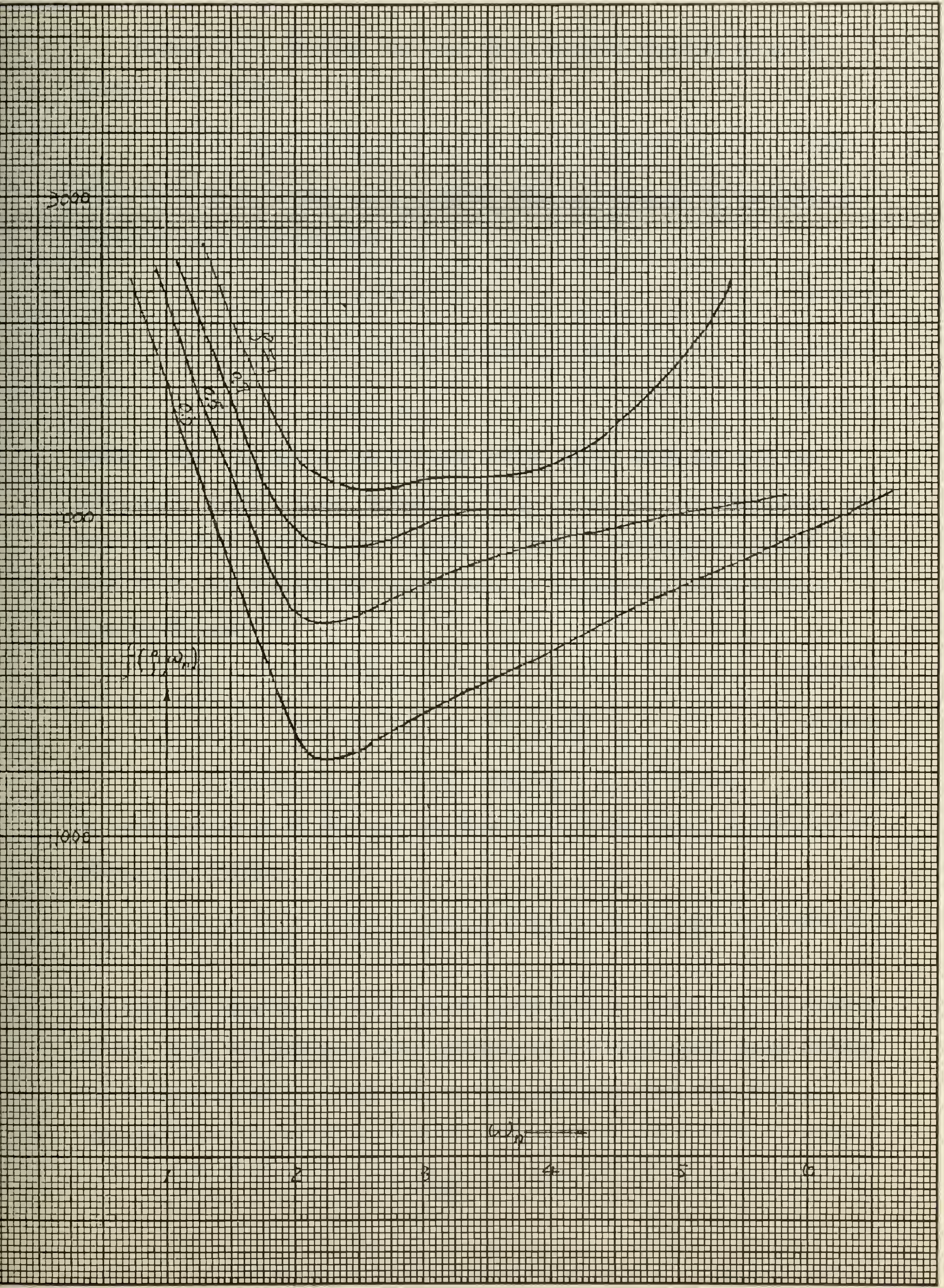
As the coefficients D's are functions of the arbitrarily chosen roots and the definition of 'a', the inequality from Routh's criterion defines a region on the S-plane directly.

The coefficients D's for a nth order reduced characteristic equation and its Routh's criterion inequality have been formulated and tabulated in Table 2-4. Since the reduced characteristic equation of any order system and any number of variables can be derived, the process of evaluating the dominant root-region for the definition described above can be applied to any order system and any number of variables.

Table 2-4. Transformed coefficients D's of the reduced characteristic equation "a" is the mapping contour, C's are the coefficient in the S-plane

order of the reduced equation	transformedcoeff.'s	Routh's criterion
2rd	$D_2 = 1$ $D_1 = C_1 - 2a$ $D_0 = C_0 - C_1 a + a^2$	$D_1 > 0$ $D_0 > 0$
3rd	$D_3 = 1$ $D_2 = C_2 - 3a$ $D_1 = C_1 - 2aC_2 + 3a^2$ $D_0 = C_0 - C_1 a + a^2 c^2 - a_3^3$	$D_1 > 0$ $D_2 > 0$ $D_1 D_2 > D_0$
4th	$D_4 = 1$ $D_3 = C_3 - 4a$ $D_2 = C_2 - 3aC_3 + 6a^2$ $D_1 = C_1 - 2aC_2 + 3a^2 C_3 - 4a^3$ $D_0 = C_0 - aC_1 + a^2 C_2 - a^3 C_3 + a^4$	
n th	$D_n = C_n$ $D_{n-1} = C_{n-1} - naC_n$ $D_{n-2} = C_{n-2} - (n-1)aC_{n-1} + \frac{n(n-1)}{1\cdot 2}a^2 C_n$ $D_{n-3} = C_{n-3} - (n-2)aC_{n-2} + \frac{(n-1)(n-2)}{1\cdot 2}a^2 C_{n-1} - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^3 C_n$ $D_{n-4} = C_{n-4} - (n-3)aC_{n-3} + \frac{(n-2)(n-3)}{1\cdot 2}a^2 C_{n-2} - \frac{(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3}a^3 C_{n-1} + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}a^4 C_n$ $\vdots$	

Fig. 2-15 Plot of inequality 2-37.



Example 2-2. Consider the same characteristic equation given in Example 2-1, namely -

$$s^5 + B_4 s^4 + B_3 s^3 + f_2 s^2 + f_1 s + B_o = 0 \tag{2-34}$$

where $B_4 = 16$, $B_3 = 128$, and $B_o = 2000$. The coefficients of the reduced characteristic equations are:

$$C_2 = 16 - 2\zeta\omega_n \tag{2-35-1}$$

$$C_1 = 128 - 32\omega_n + (4\zeta^2 - 1)\omega_n^2 \tag{2-35-2}$$

$$C_o = 2000/\omega_n^2 \tag{2-35-3}$$

where ζ and ω_n define the dominant roots, since there are only two arbitrarily chosen roots. Assume $a = \zeta\omega_n$, i.e., all the other roots must have real parts greater than or equal to $\zeta\omega_n$, Substitute equation (2-35) into (2-31) the transformed coefficients of the reduced characteristic equation become:

$$D_2 = 16 - 5\zeta\omega_n \tag{2-36-1}$$

$$D_1 = 128 - 64\zeta\omega_n + (11\zeta^2 - 1)\omega_n^2 \tag{2-36-2}$$

$$D_o = 2000/\omega_n^2 - 128\zeta\omega_n + 48\zeta^2\omega_n^2 - \zeta(7\zeta^2 - 1)\omega_n^3 \tag{2-36-3}$$

By substitution of (2-36) into the Routh's criterion $D_1 D_2 > D_o$ and manipulating, one obtains:

$$1536\zeta\omega_n - 16(28\zeta^2 - 1)\omega_n^2 - 4\zeta(1 - 12\zeta^2)\omega_n^3 + B_o/\omega_n^2 < 2048 \tag{2-37}$$

Let the left side be $f(\zeta, \omega_n)$, choose ζ and plot f versus ω_n as shown in Fig. 2-15, the dominant roots (defined by $a = \zeta\omega_n$) region which satisfy the definition for $\zeta = 0.3$, 0.5, 0.7 are evaluated from Fig. 2-15 as follows:

$$\zeta = 0.3 \qquad 1.4 \le \omega_n \le 6.3$$

$$\zeta = 0.5 \qquad 1.65 \le \omega_n \le 5$$

$$\zeta = 0.7 \qquad 1.9 \le \omega_n \le 4.8$$

This region is plotted in the S-plane as shown in Fig. 2-16, in which the stability boundary is also plotted.

In the dominant root-region, assume the dominant roots are chosen to be $-S_1, -S_2 = 1.89 \pm j\,6$; $(\zeta = 0.3,\ \omega_n = 6.3)$

the secondary roots evaluated from the reduced characteristic equation are:

$$-S_3, -S_4 = -2.8 \pm j\,1.0, \quad (\zeta = 0.94, \quad \omega_n = 2.82)$$

$$-S_5 = -6.6$$

It shows that the secondary roots have real parts which are greater than that of the dominant roots.

If the dominant roots are chosen as

$$-S_1, -S_2 = 2.5 \pm j\,4.32, \quad (\zeta = 0.5,\ \omega_n = 5)$$

the secondary roots are:

$$-S_3, -S_4 = 3.56 \pm j\,2.76, \quad (\zeta = 0.79,\ \omega_n = 4.5)$$

$$-S_5 = -3.88$$

It shows again the real parts of the secondary roots are greater than that of the dominant roots.

In the above two choices of the dominant roots, all the secondary roots also have greater damping ratio ζ. However the dominant root region evaluated by this procedure doesn't imply this restriction to the secondary roots, but this restriction can be obtained by choosing large value of "a" relative to $\zeta\omega_n$ and by inspection of the shape of the region.

In Fig. 2-16, it can be seen that the dominant root-region is inside the stability root-region. The secondary roots are not necessarily inside

the stability boundary. The max. damping of the dominant roots can also be evaluated from the dominant root-region.

Example 2-3. Given a 6th order characteristic equation

$$S^6 + B_5 S^5 + B_4 S^4 + f_3 S^3 + f_2 S^2 + f_1 S + B_0 = 0$$

where f_1, f_2, f_3 are adjustable, $B_5 = 184$, $B_4 = 10840$ and $B_o = 3x10^8$.

Here three coefficients are adjustable, three roots can be chosen arbitrarily. Assume the requirements of the dominant roots is such that all the secondary roots must have the real negative parts at least twice that of the dominant roots, then the mapping contour is $a = 2\zeta\omega_n$. In order to meet the requirements of the secondary root, the third arbitrarily chosen root must be set at least to the min. value, namely $S_3 \geq 2\zeta\omega_n$. Assume $S_3 = 2\zeta\omega_n$ then the three arbitrarily chosen roots are:

$$-S_1, -S_2 = -\zeta\omega_n \pm j\,\omega_n\sqrt{1-\zeta^2} \quad \text{- - - - -} \quad \text{dominant}$$

$$-S_3 = -2\zeta\omega_n$$

From Table 1-5, the coefficients of the reduced characteristic equation are:

$$C_2 = B_5 - (S_1 + S_2 + S_3) \tag{2-37-1}$$

$$C_1 = B_4 - C_2(S_1 + S_2 + S_3) - (S_1S_2 + S_2S_3 + S_3S_1) \tag{2-37-2}$$

$$C_0 = B_0 / S_1 S_2 S_3 \tag{2-37-3}$$

Substitute S_1, S_2 and S_3 into (2-37) and manipulate, obtain

$$C_2 = B_5 - 4\zeta\omega_n \tag{2-38-1}$$

$$C_1 = B_4 - 4\zeta B_5\omega_n + (12\zeta^2 - 1)\omega_n^2 \tag{2-38-2}$$

$$C_0 = B_0 / 2\zeta\omega_n^3 \tag{2-38-3}$$

Fig. 2-16. Dominant root region of a 5th order equation with f_1, f_2 variable.
Shaded area: dominant root region Dash line: Boundary of stability.

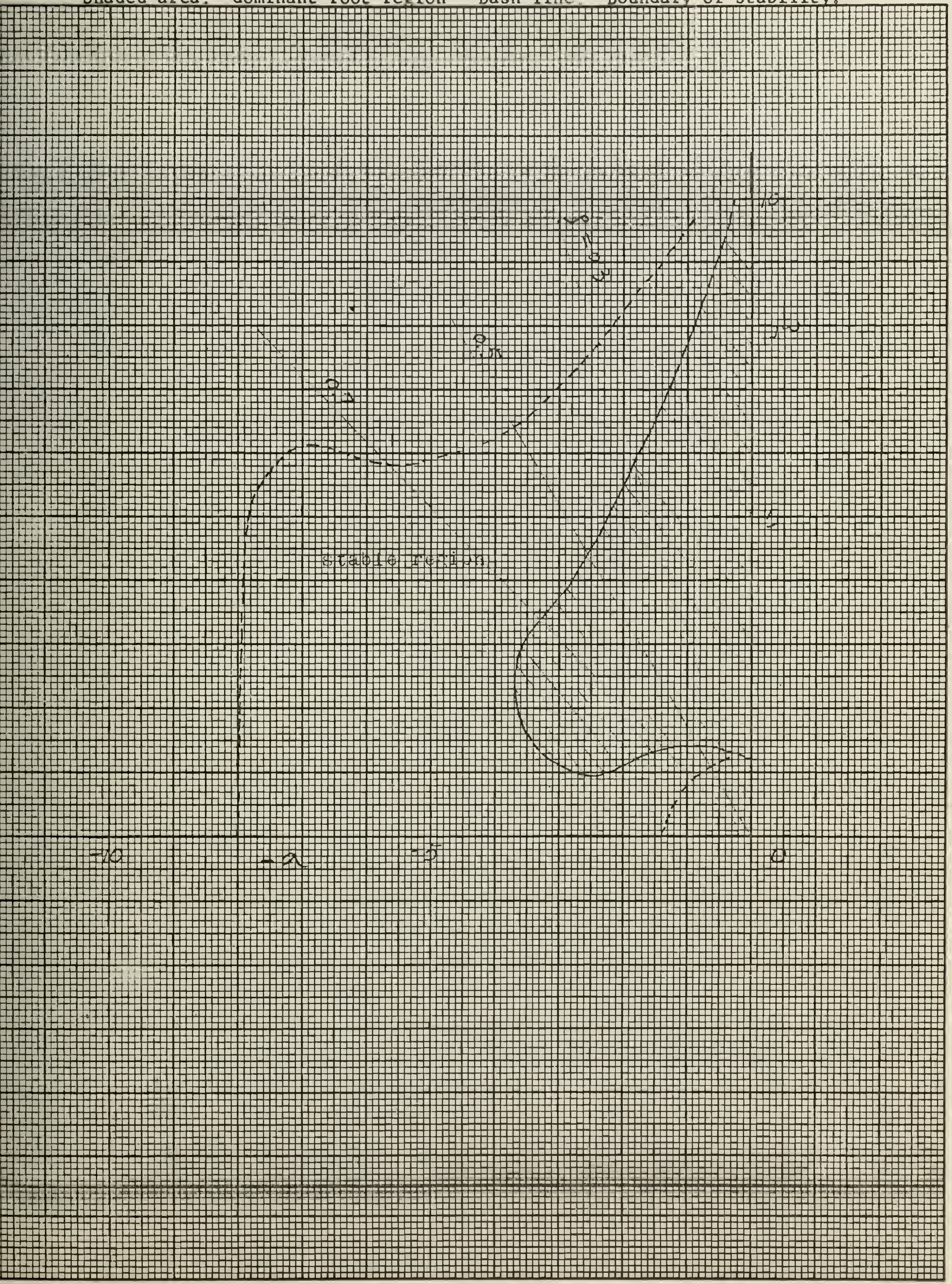
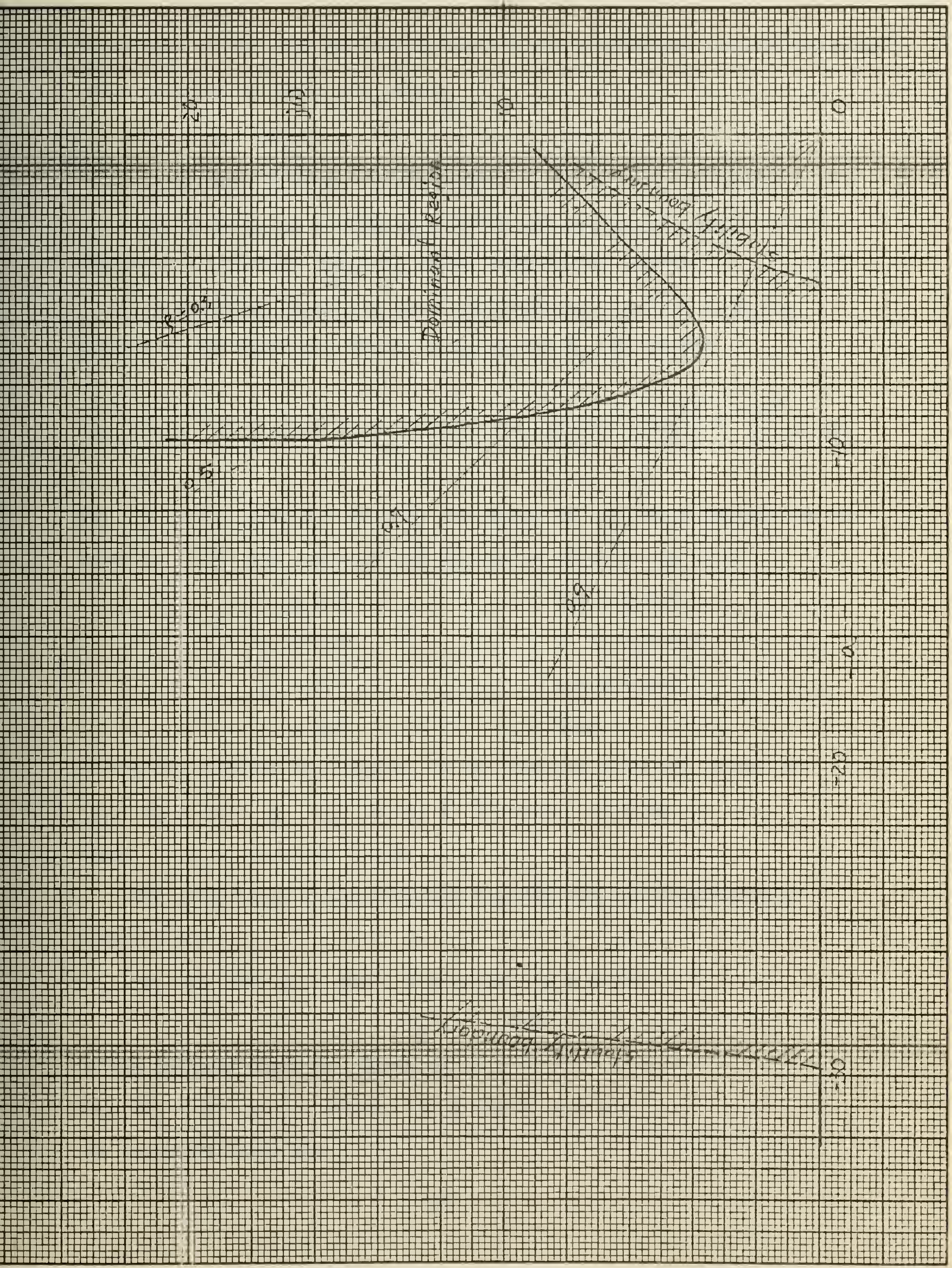


Fig. 2-17. Dominant Root Regions of Example 2-3.



Substitute (2-38) into the coefficients of the transformed reduced characteristic equation in Table 2-4, obtain

$$D_2 = B_5 - 10\zeta\omega_n$$

$$D_1 = B_4 - 8\zeta B_5 \omega_n + (40\zeta^2 - 1)\omega_n^2$$

$$D_0 = B_0/2\zeta\omega_n^3 - 2\zeta B_4 \omega_n + 12\zeta^2 B_5 \omega_n^2 - 2\zeta(24\zeta^2 - 1)\omega_n^3$$

The Routh's criterion $D_2 D_1 > D_0$ after manipulation becomes

$$\omega_n^3 \left[B_4 B_5 - 8\zeta(B_5^2 + B_4)\omega_n + B_5(108\zeta^2 - 1)\omega_n^2 - 8\zeta(44\zeta^2 - 1)\omega_n^3 \right] > B_0/2\zeta$$

Substitute the numerical values of B_5, B_4 and B_0, it becomes

$$\omega_n^3 \left[1.56\times10^6 - 8\zeta(4.2)\times10^4\omega_n + 184(108\zeta^2 - 1)\omega_n^2 - 8\zeta(44\zeta^2 - 1)\omega_n^3 \right] > \frac{1.5\times10^8}{\zeta} \quad (2\text{-}39)$$

For ζ = 0.3, 0.5, 0.7 and 0.9, the region of ω_n which satisfy the inequality (2-39) are evaluated and plotted as shown in Fig. 2-17.

$\zeta = 0.3$	$7.5 \leq \omega_n \leq 30$
$\zeta = 0.5$	$7 \leq \omega_n \leq 19.5$
$\zeta = 0.7$	$6.8 \leq \omega_n \leq 13$
$\zeta = 0.9$	None

Within the dominant root region, if the dominant roots are chosen as:

$-S_1$, $-S_2$ = $-9.5 \pm j\,16.4$; ($\zeta = 0.5$, $\omega_n = 19$)

then: $-S_3 = -2\zeta\omega_n = -19$

The other three roots evaluated from the reduced characteristic equation are:

$-S_4$, $-S_5 \doteq -20 \pm j25$; ($\zeta = 0.97$, $\omega_n = 20.6$)

$-S_6 = -106$

If the dominant roots are chosen as

$-S_1$, $-S_2$ = $-8 \pm j13.8$, ($\zeta = 0.5$, $\omega_n = 16$)

The secondary roots are

$-S_3 = -16$

$-S_4, -S_5 = -23 \pm j\,13, \quad (\zeta = 0.87, \ \omega_n = 26.4)$

$-S_6 = -106$

The evaluation of the dominant root region is not necessarily carried out point by point, only those points of interest are evaluated. And furthermore, only the boundary points which satisfy the Routh's criterion are of significance. The Routh's criterion formed in this process is a polynomial of ω_n in any case, therefore for higher order reduced characteristic equations there may be more than one region which satisfies the criteria. But only the region which is closest to the $j\omega$ axis is of interest, because the roots are assumed to be dominant.

2-6. Modified Dominant Root Region - One Zero Close to the Origin.

The assumption that the response of a system can be characterized by a pair of complex dominant roots is based on the approximations that all other roots are either too far to the presumable dominant roots or close to a zero. Consider Fig. 2-18, the roots $(-S_1)$ and $(-S_2)$ are said to be dominant

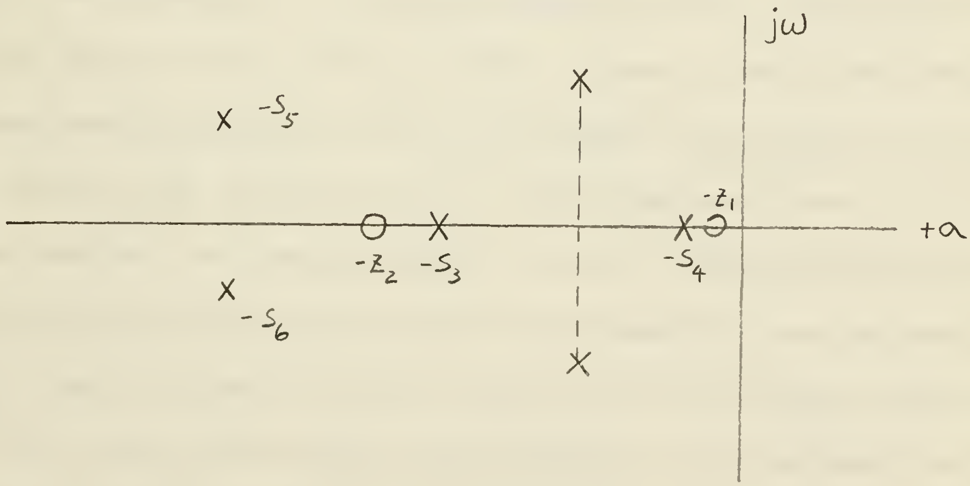


Fig. 2-18 General Closed Loop Pole-zero Configuration.

because the other roots ($-S_5$), ($-S_6$) and ($-S_3$) are far to the left and ($-S_4$) is close to a zero ($-Z_1$) so that they contribute negligibly to the transient response. If no zeros lie to the right of the dominant roots (close to the origin), then all the secondary roots are required to be far to the left if the presumed dominant roots are dominant. If there is a zero close to the origin, then one root to the right of the dominant roots has negligible effect on the dominance if that root is close enough to the zero. Therefore, the dominant-root region defined in the last section must be modified when there are zeros close to the origin. Usually, the zeros of a system are known. If the known zero is far to the left of the dominant roots, then the dominant root region defined in the last section is still valid. When the zero is known to be close to the origin, the modification of the dominant root region is required.

Consider Fig. 2-19(a) which is a third order system with a cascade compensator. Assume the compensator is a lag network. Let ($-S_1$) and ($-S_2$) be the arbitrary roots (also the dominant roots in this case), then (Z) and (p) are functions of S_1 and S_2 (Equation 1-22). The general pattern of the root locus is shown in Fig. 2-19(b). Since S_1 and S_2 are arbitrary roots, then from the tendency of the root locus, the one only root which is possible close to the origin is ($-S_3$). If the arbitrary roots are chosen within the dominant root region (point B in Fig. 2-19E) defined in the last section and the pole (p) and zero (Z) of the compensator are adjusted accordingly, then ($-S_3$) and ($-S_4$) are to the left of the dominant roots (Fig. 2-19b). If S_1 and S_2 are chosen on the boundary of the dominant root region (Point C in Fig. 2-19E), then the root ($-S_3$) is forced to be in a position as shown in Fig. 2-19C. As the arbitrary roots move toward the

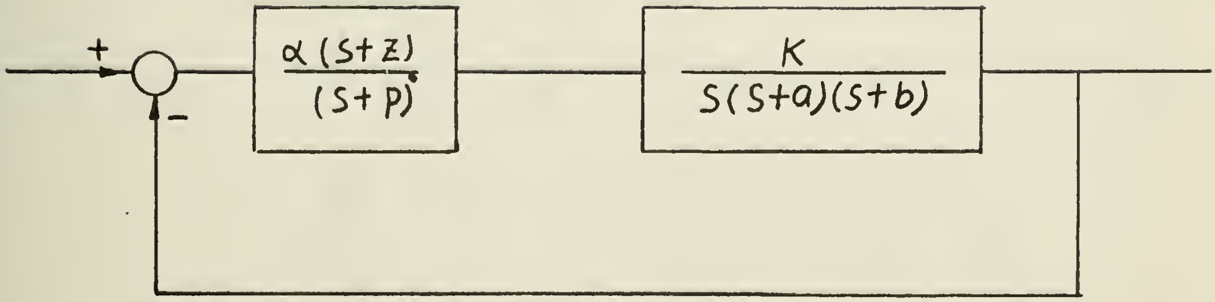


Fig. 2-19a Cascade filter.

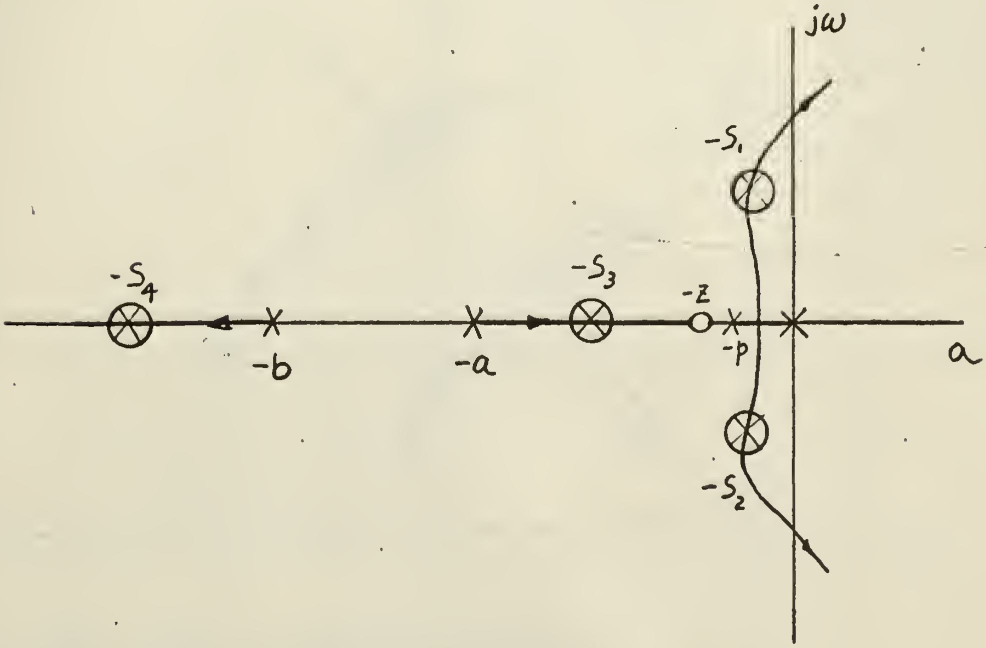


Fig. 2-19b Root locus of Fig. 2-19(a) in case Z > p

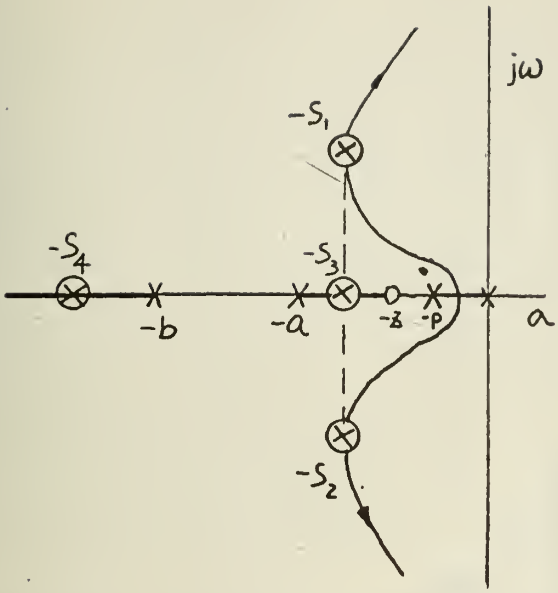


Fig. 2-19C Dominant roots on the boundary of the dominance for $a = \zeta\omega_n$

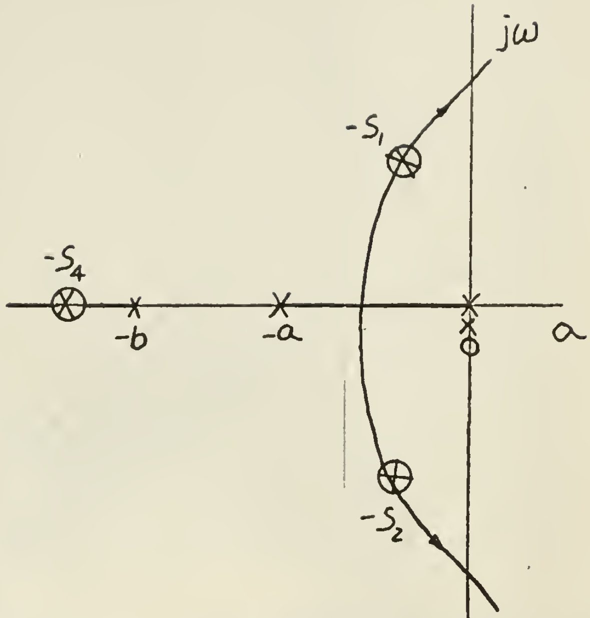


Fig. 2-19D Dominant roots on the boundary of stability

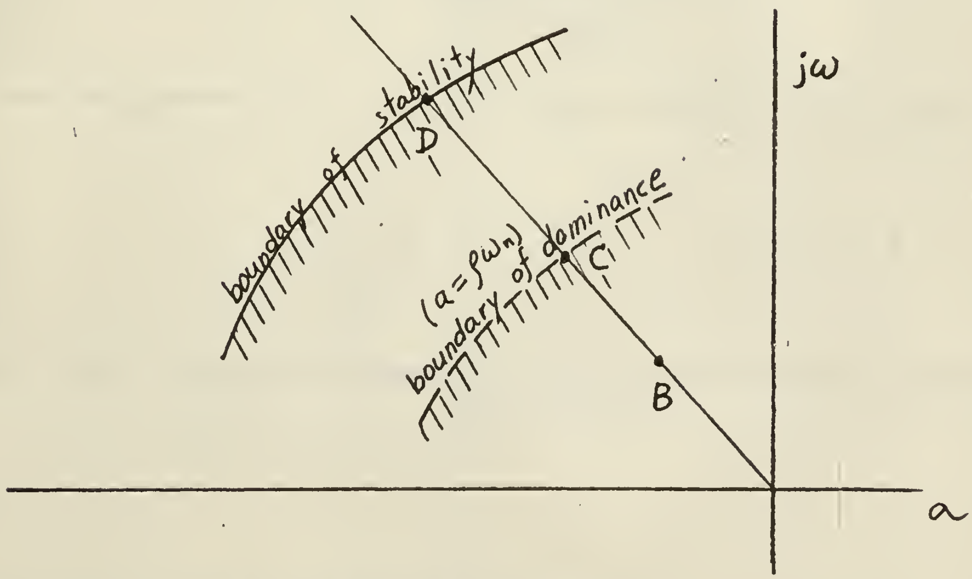


Fig. 2-19E Stable and dominant root-region.

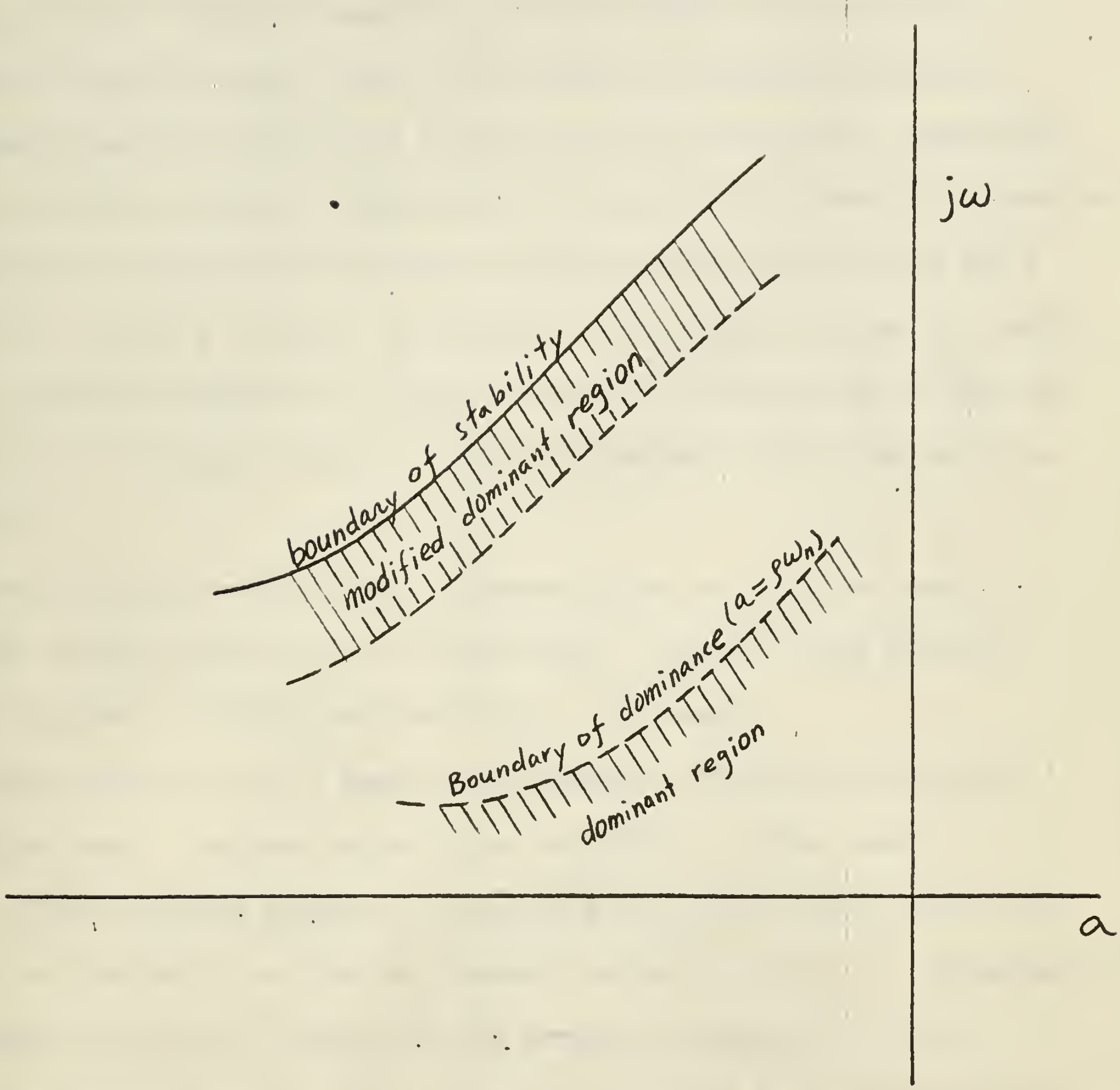


Fig. 2-20 Dominant root region ($a = \zeta\omega_n$) and Modified Dominant Root Region

origin and so does the zero. When the arbitrary roots reach the boundary of stability (Point D in Fig. 2-19E), ($-S_3$) reaches the origin and the zero and pole of the compensator also reach the origin as shown in Fig. 2-19D. From this analysis, it can be visualized that if the dominant roots are chosen close to the stability boundary and within the stable region, this implies a dipole close to the origin. Therefore, a region which is close to the stability boundary and between the boundaries of stability and dominance (which are defined in the last section for $a = \zeta\omega_n$) is also a dominant root region. This region is given the terminology "modified dominant root region" in order to distinguish it from the definition of the last section. These two dominant regions are shown in Fig. 2-20.

The pole-zero configuration discussed above is often the case for systems of high velocity error constant (K_v). For very high velocity error constant, a dipole near the origin is necessary.

The analysis above is based on the tendency of the root-locus and the known zero. The same analysis can be applied to other cases.

In this case the dominant roots should be chosen either close to the stability boundary or within the dominant region ($a = \zeta\omega_n$). Choosing the dominant roots in the vicinity of the dominant boundary ($a = \zeta\omega_n$) should be avoided when a zero is close to the origin because the root is not close enough to the zero and it may cause excessive overshoot. If a zero is known to the right of the dominant roots, a root close to it is necessary.

Now consider the pole-zero configuration of Fig. 2-19D which is the situation of stable limit. In this condition, the pole and zero of the compensator are at the origin, therefore the presumed dominant roots $-S_1$

and $-S_2$ are on the original uncompensated root-locus (K as variable). This provides another means to evaluate the stable boundary, namely the stable boundary is the uncompensated root-locus itself in this case (lag compensator). This boundary line is one of the dividing lines of the root relocation zone discussed in reference 7. For high order systems there are several branches of the stable boundary line, but for the purpose of dominant region only the one which is closest to the $j\omega$ axis is of interest. From the root-locus it can be visualized easily that the dominant roots are within the region which is the closest to the $j\omega$ axis. From this analysis, it concludes that for lag compensator the region closest to the nearest branch of the uncompensated root-locus is the dominant root region. And the dominant boundary of the definition $a = \zeta\omega_n$ provides a guide of how close they should be.

Example 2-4. In Fig. 2-21, K_v is not to be reduced. Find the dominant and modified dominant root region:

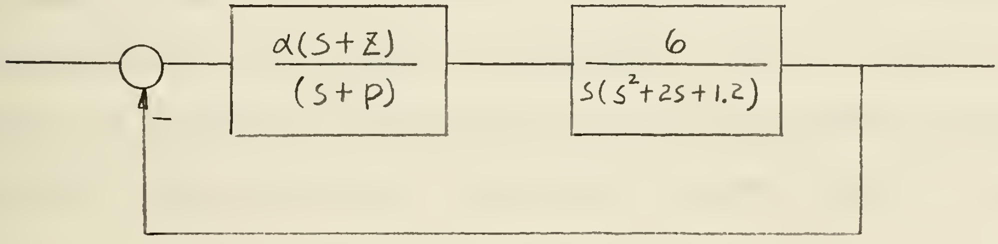


Fig. 2-21 Example 2-4.

The expressions for the dependent variables Z, p, C_o and C_1 have been derived in (1-22) and (1-23). Substitute $S_1 + S_2 = 2\zeta\omega_n$, $S_1S_2 = \omega_n^2$, $B_o = 6$, $B_1 = 1.2$, $B_2 = 2$ into (1-22) and (1-23), one obtains

$$C_o = [(1-4\varsigma^2)\omega_n^2 - 1.2 + 4\varsigma\omega_n]6/(2\omega_n^2 - 2\varsigma\omega_n^3 - 6) \tag{2-40-1}$$

$$p = \omega_n^2 C_o/6 \tag{2-40-2}$$

$$C_1 = 2 + p - 2\varsigma\omega_n \tag{2-40-3}$$

$$\alpha = [\omega_n^2 C_1 + 2\varsigma\omega_n C_o - 1.2p]/6 \tag{2-40-4}$$

From Table 2-4:

$$D_1 = C_1 - 2a$$

$$D_o = C_o - C_1 a + a^2$$

For a = $\varsigma\omega_n$

$$D_1 = C_1 - 2\varsigma\omega_n \tag{2-41-1}$$

$$D_o = C_o - \varsigma\omega_n C_1 + \varsigma^2\omega_n^2 \tag{2-41-2}$$

The Routh's criterion for D_1 and D_o is $D_1 > 0$ and $D_o > 0$.

The stability boundary line (root-locus of the uncompensated system) is shown in Fig. 2-22a. In order to evaluate the a = $\varsigma\omega_n$ dominant boundary line, C_o and C_1 must be calculated. For $\varsigma = 0.3$ and $\varsigma = 0.5$, the results of computation from equations (2-40) are tabulated in Table 2-5a and Table 2-5b respectively. The dominant boundary line for a = $\varsigma\omega_n$ is

Table 2-5a. Example 2-4 for $\varsigma = 0.3$

ω_n	C_o	C_1	p	α	z	D_1	D_o
0.4	0.65	1.777	0.01735	0.0698	0.0249	+	+
0.5	0.473	1.72	0.0197	0.0914	0.216	+	+
0.6	0.278	1.657	0.01662	0.118	0.141	+	+
0.7	0.0539	1.584	0.0044	0.1322	0.0335	+	-
0.8	-0.085						

Table 2-5b Example 2-4 for $\zeta = 0.5$

ω_n	C_0	C_1	P	α	Z	D_1	D_0
0.2	0.81	1.8	0.0054	0.0543	0.0995	+	+
0.3	0.615	1.7	0.00923	0.0545	0 1695	+	+
0.4	0.418	1.611	0.01115	0.0685	0.1625	+	+
0.5	0.213	1.509	0.00886	0.079	0.112	+	-
0.6	0						

Shown in Fig. 2-22a. If the dominant roots $\zeta = 0.5$, $\omega_n = 0.55$ are chosen which is close to the stability boundary, the closed loop pole-configuration is shown in Fig. 2-22b. For this choice of the dominant roots, the compensator is

$$G_c = \frac{0.084(s+0.066)}{s+0.00543}$$

Fig. 2-22a Dominant root region of Example 2-4. Shaded area is the modified dominant root region.

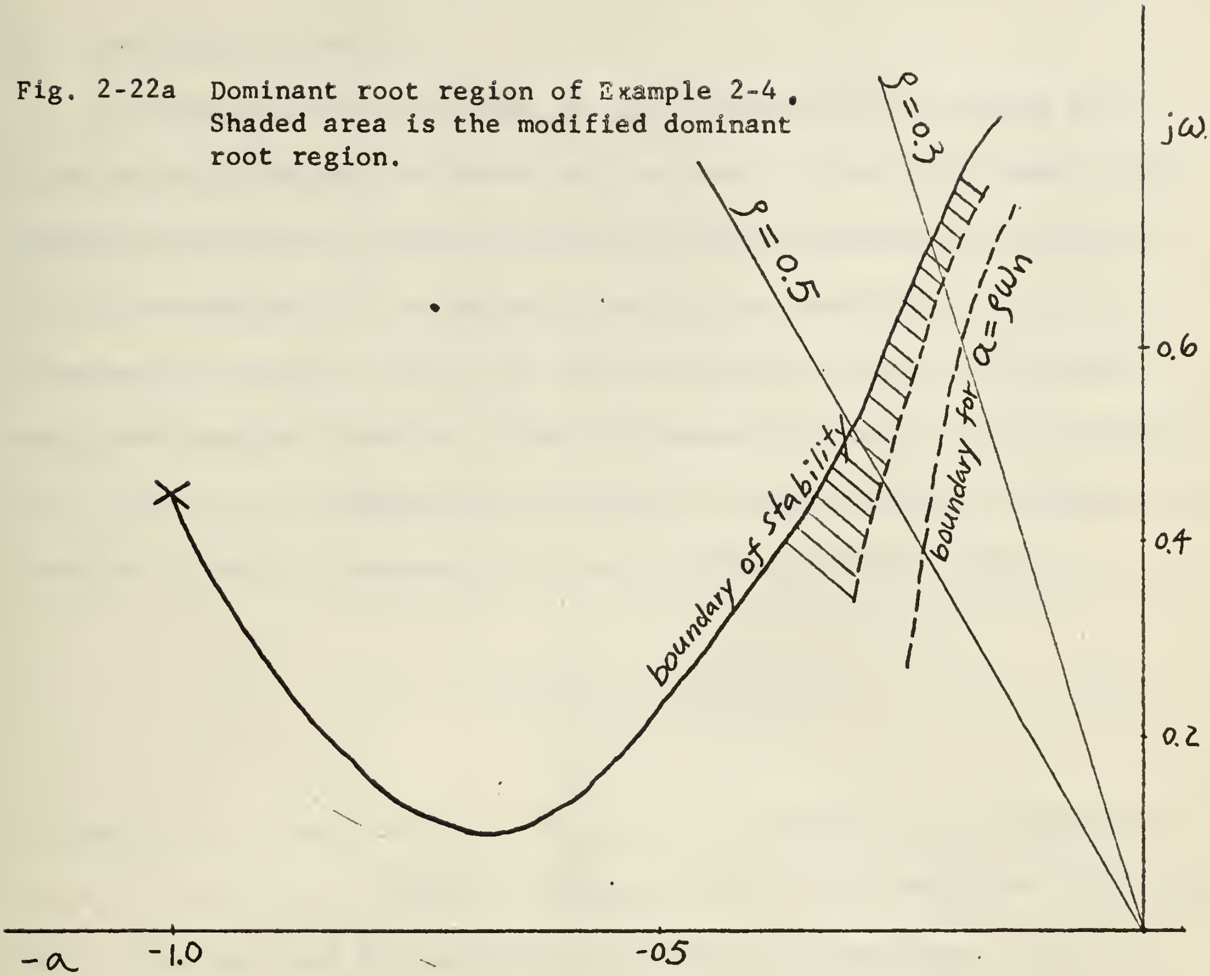
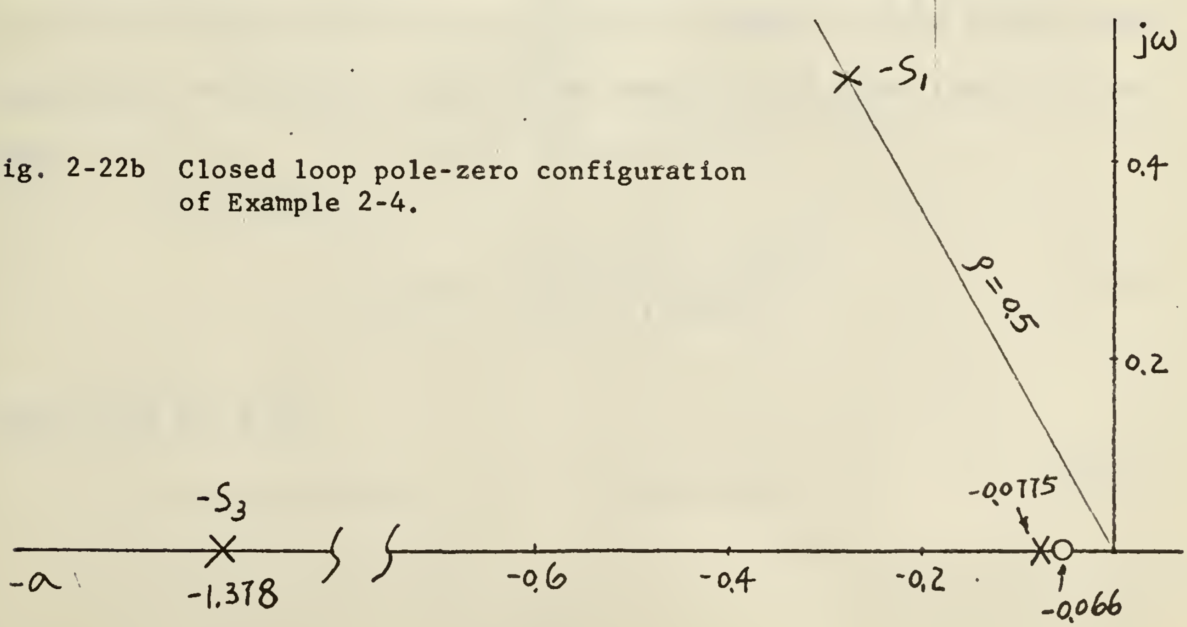


Fig. 2-22b Closed loop pole-zero configuration of Example 2-4.



CHAPTER III

FUNCTION OF VARIABLE PARAMETERS

3-1 Realization Problem:

The result of the partition process employed in this method is to separate the dependent variables into two parts: the coefficient of the reduced characteristic equation and the variable parameters (parameters of the compensator). The design is carried out essentially by two independent procedures. One is to choose appropriate closed loop pole-zero configuration from the reduced characteristic equation and the other is to realize the compensator according the chosen pole-zero configuration. Consider a cascade compensator of the following transfer function.

$$G_c = \frac{K_c(s^2 + a_1 s + a_o)}{(s^2 + b_1 s + b_o)} \tag{3-1}$$

If there are no restrictions to the type of compensator, the parameters a_1, a_o, b_1 and b_o are linearly dependent in the root-coefficient relations. Their solutions from the partition functions are always real. For the function of (3-1) with all real coefficients always can be realized either by a passive network or an active network. However, if the poles of the compensator are required to be on the negative real axis, then (3-1) becomes

$$G_c = \frac{K_c(s^2 + a_1 s + a_o)}{(s + P_1)(s + P_2)} \tag{3-2}$$

From (3-2) and (3-1)

$$P_1 + P_2 = b_1 \,, \qquad P_1 P_2 = b_o \tag{3-3}$$

In equation (3-3), P_1 and P_2 are quadratic functions. For certain choices of the roots the solution of P_1 and P_2 may be complex numbers, although b_1 and b_0 are always real. Therefore the choice of the arbitrary roots are not only necessary in the dominant root region but also necessary in the region which gives a physically realizable compensator. For certain kinds of specifications and restrictions, the compensator may be forced to be an active network[8], but for most of the systems, the designer should try a passive network first. The same realization problem may arise from feedback compensators and even pure derivative feedback.

3-2 Linear Dependence.

If the variable parameters are linearly dependent in the coefficients of the controlled characteristic equation, then the compensators are physically realizable passive or active for every choice of the arbitrary roots, because solutions of them are always real numbers. For this reason, the variable parameters in the coefficients of the controlled characteristic equation should be arranged (as possible) such that they are linearly dependent. In many cases, the linear dependent relations can be obtained by transformations. Consider a single section cascade compensator

$$G_c = \frac{K_c(s+z)}{(s+p)} \tag{3-4}$$

If K_c, z and p are variable parameters then the term $K_c z$ is a quadratic term. However, by defining $K_c z = \alpha$ then (3-4) becomes

$$G_c = \frac{K_c s + \alpha}{s+p}, \qquad \text{where } K_c = \alpha/z \tag{3-5}$$

For the transfer function of (3-5) alone the parameters are linearly dependent. For the same reason a single section compensator in the feedback path the relations can be linearized in the same way.

The linear dependence of the variable parameter is a sufficient condition for the compensator to be realizable, but it is not necessary. The

necessary and sufficient condition can be found. For example, a single section cascade compensator, the roots within the single section relocation zones formed by the rules of reference (7) are the necessary and sufficient condition.

Realization problems may even arise for pure derivative feedback. Consider Fig. 3-1 in which the point "X" is assumed to be able to insert a summer and the signal at which can be measured. Assumed pure derivative feedback is devised as shown in the Fig. 3-1. From rules of signal flow graph the determinate of the system contains a term which is the product of the two nontouching loop transmissions. This term contains the

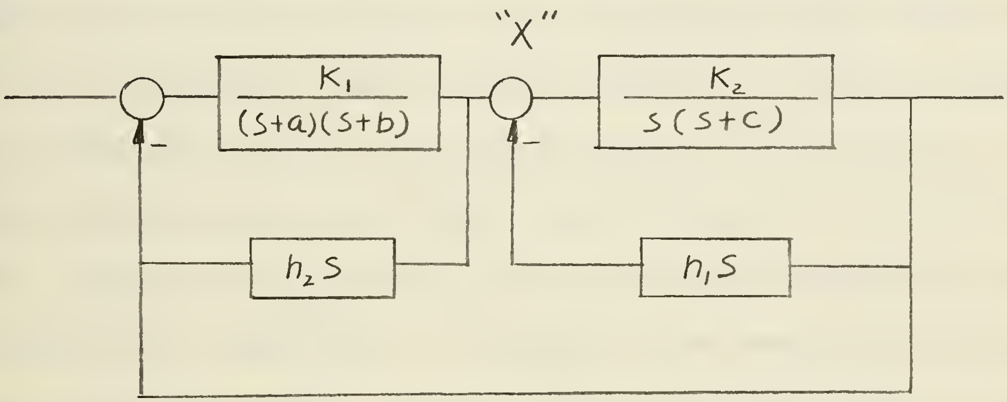


Fig. 3-1. Pure Derivative Feedback

product of h_1 and h_2. For a certain choice of the roots, h_1 and h_2 may be complex numbers. For some purposes in control systems, the system may be expected to be overdamped and to operate in a given set of roots such as discontinuous operation in reference 9. For that purpose the scheme of Fig. 3-1 should be avoided. For pure derivative feedback to achieve a given set of roots, certain rules and methods have been worked out. They are discussed in detail in Appendix I.

3-3 Choice of Arbitrary Roots.

Any root may be chosen as the arbitrary root. For convenience the dominant roots are usually chosen as the arbitrary roots. If there is only one arbitrary root the damping ratio ζ is the convenient independent variable. If the arbitrary roots are more than two, two of them are chosen as the dominant roots, others may be either fixed at a certain value or be defined to have a certain relation with the dominant roots. Suppose the original system has a zero which is to the right of the expected dominant roots. Then in the case of three arbitrary roots one of them may be chosen to cancel that zero or be close to it. In general, the more the arbitrary roots, the more the design freedom one has. The forward gain K may be always treated as a variable parameter, so one additional arbitrary root may be introduced even though K isususally specified within some limits. By treating K as a variable parameter, sometimes it may simplify the relations, but sometimes may make the problem more complicated. It depends on the specifications and the nature of the compensator, but the increase of design freedom is always true.

For the case when a large number of arbitrary roots i.e., a large number of variable parameters is involved, a clear picture of the closed loop pole-zero configuration must be assumed. With that pole-zero configuration the arbitrary roots can be properly related to the dominant roots. In the mapping process from the contour of the arbitrary roots to the variable parameters, only one independent variable is varied, i.e., the parametrical line is obtained in the parametrical space (or coefficient space of the reduced characteristic equation). There are infinitely many parametrical lines in the parametrical space, but only those which are within

the specification limits are of interest.

3-4 Coefficient as Variables: Mitrovic's Method.

In case of coefficients as variables, this method provides a relatively simple relation. For lower order systems and for derivative feedback the application is direct. For the case of any two coefficients taken as variables, the function of the variable coefficients are the same as Mitrovic's coefficient functior which has a certain interpretation from the mapping process. In order to show the coefficient functions are the same as that of Mitrovic's method, take a 5th order equation with B_0 and B_1 as variables from Table 1-4, and make substitution of $S_1+S_2 = 2\rho\omega_n$ and $S_1S_2 = \omega_n^2$ from which one obtains:

$$C_2 = B_4 - 2\rho\omega_n$$

$$C_1 = B_3 - 2\rho\omega_n [B_4 - 2\rho\omega_n) - \omega_n^2$$

$$= B_3 - 2\rho B_4\omega_n + (4\rho^2-1)\omega_n^2$$

$$C_0 = B_2 - 2\rho B_3\omega_n + (4\rho^2-1)B_4\omega_n^2 - [2\rho(4\rho^2-1) - 2\rho]\omega_n^3$$

$$f_0 = \omega_n^2[B_2 - 2\rho B_3\omega_n + (4\rho^2-1)B_4\omega_n^2 - 2\rho(4\rho^2-1)\omega_n^3] \quad (3\text{-}6)$$

$$f_1 = 2\rho\omega_n[B_2 - 2\rho B_3\omega_n + (4\rho^2-1)B_4\omega_n^2 - 2\rho(4\rho^2-1)\omega_n^3] + [B_3 - 2\rho B_4\omega_n + (4\rho^2-1)\omega_n^2]\omega_n^2$$

$$= 2\rho B_2\omega_n + (1-4\rho^2)B_3\omega_n^2 + 2\rho(4\rho^2-1)B_4\omega_n^3 + (4\rho^2-1)\omega_n^4 \quad (3\text{-}7)$$

Equations (3-6) and (3-7) are the same expressions as obtained from Mitrovic's method[2]. By the same manipulation, functions from Table 1-1 to Table 1-3

are the same expressions as obtained from the modified Mitric's method[3]. Mitrovic's original coefficient function is derived by mapping a constant ζ contour through the characteristic equation. By the same mapping process, other contours also can be used. In this analytical method the functions are expressed in terms of complex variables, the functions for any contour can be obtained readily. Because of this extension of Mitrovic's Method to the contour other than constant ζ line, some of the features are discussed briefly in the next three sections.

3-5 Cauchy's Theorem Applied to the Function of Variable Coefficients.

In the case of two variable coefficients of a characteristic equation, the interpretation of the parametrical line of f_i vs f_j is discussed in detail in reference (2) and reference (4). Here only the cases of variable coefficients which are not in sequence are discussed. Take the case of f_1 and f_3 as variable coefficients. Assume the original characteristic equation is

$$F = S^n + \cdots + B_3 S^3 + B_2 S^2 + B_1 S + B_0 \qquad (3\text{-}8)$$

and assume the controlled characteristic equation is

$$f = S^n + \cdots + f_3 S^3 + B_2 S^2 + f_1 S + B_0 \qquad (3\text{-}9)$$

If f_1 and f_3 are so chosen such that $f = 0$, then

$$F = (B_3 - f_3) S^3 + (B_1 - f_1) S \qquad (3\text{-}10)$$

Assume the arbitrary roots are on a constant ζ line in the left half of the complex plane, then the angle of the S^3 term of (3-10) is $\angle 3\theta$ and the angle of the S term is $\angle\theta$ if $B_3 > f_3$ and $B_1 > f_1$ respectively.

If $B_3 < f_3$ and $B_1 < f_1$, then the angle of the corresponding term is $\angle -3\theta$ and $\angle -\theta$ respectively as shown in Fig. 3-2(b). The functions of f_1 and f_3 have negative values as $\omega << 1$ from Table 1-3. Assume all the roots of equation (3-9) are to the left of the ζ line, then the mapping of ω_n from $0 \rightarrow \infty$ line, must give counterclockwise loops. At the 4 points on $F(s)$ curve as shown by (1), (2) (3) and (4), f_1 and f_3 must have the following values:

$$\begin{array}{lll} \text{at point (1)} & f_3 = B_3 & f_1 < B_1 \\ \text{at point (2)} & f_3 > B_3 & f_1 = B_1 \\ \text{at point (3)} & f_3 = B_3 & f_1 > B_1 \\ \text{at point (4)} & f_3 < B_3 & f_1 = B_1 \end{array} \qquad (3\text{-}11)$$

By tracing the curve f_1 vs f_3 by (3-11), the f_1 vs f_3 curve must have a counterclockwise loop and enclose the point M. In general, the curve f_i vs f_j in which $j > i + 1$, then a counterclockwise loop of the curve f_i vs f_j implies all the roots are to the left of the ζ line, if point M is inside the loop. In addition to the conclusions from the reference (2) and reference (4), the interpretations of f_i vs f_j curve are summarized as follows:

(1) If $j = i + 1$ ($j = 0, 1, 2, \ldots$), clockwise loop of f_i vs f_j curve implies all the roots are to the left of the ζ line providing point M is inside the loop.

(2) If $j > i + 1$ ($j = 0, 1, 2, \ldots$), counterclockwise loop of f_i vs f_j curve implies all the roots are to the left of the ζ line providing point M is inside the loop.

The above rules can also be applied when the mapping contour is a constant vertical line (α constant).

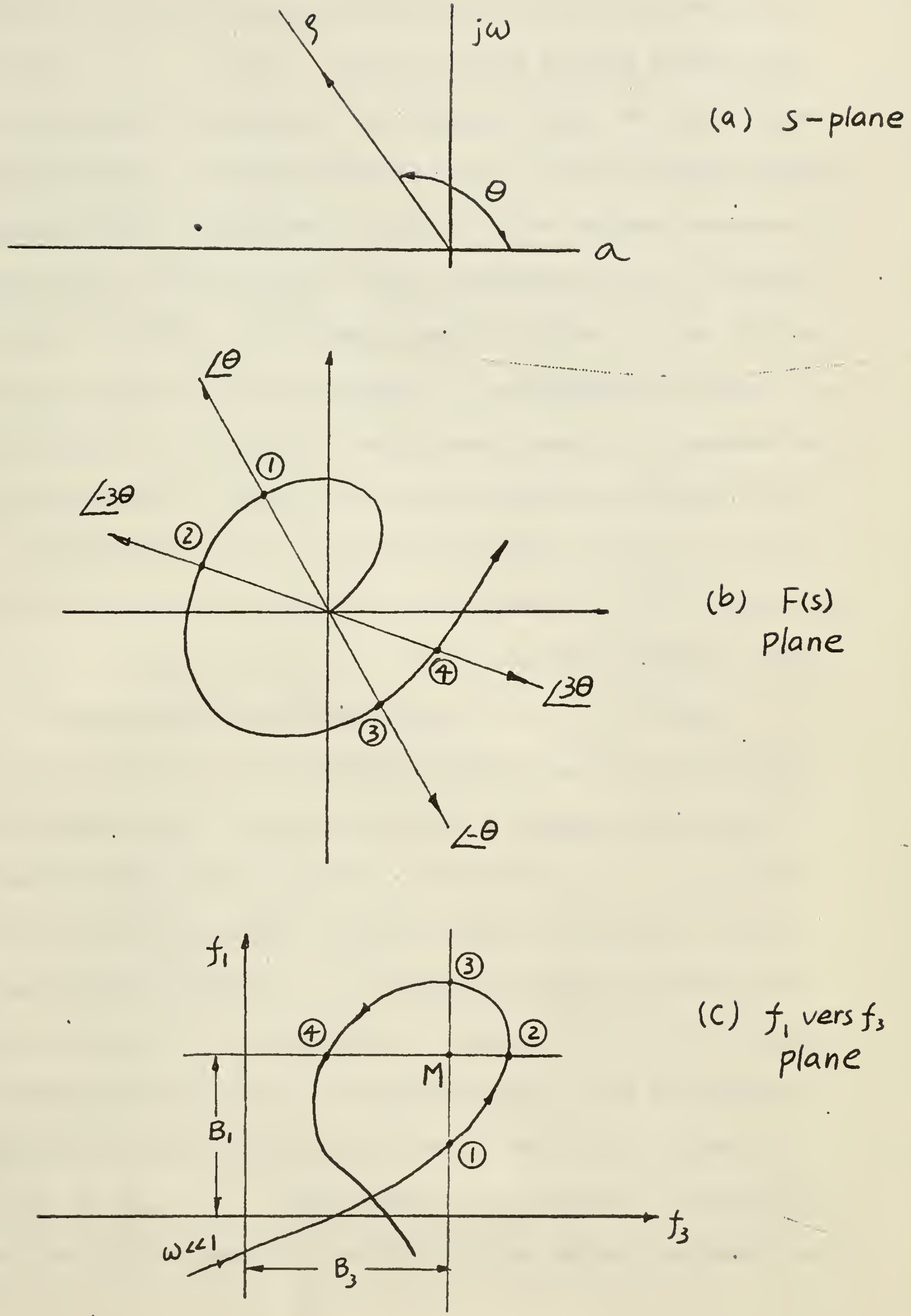
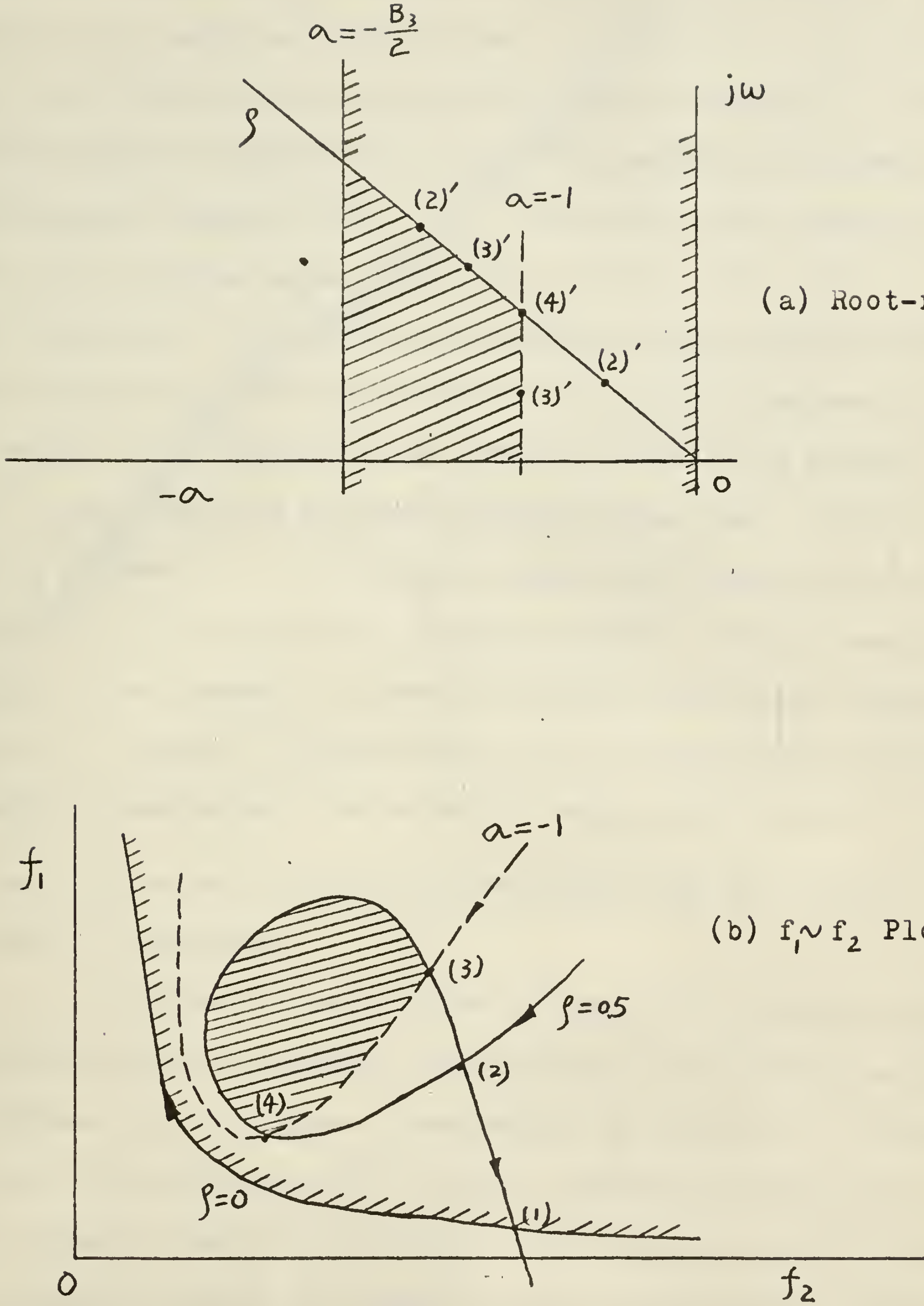


Fig. 3-2 Mapping of constant ξ contour to coefficient plane

3-6 Evaluation of roots from the f_i vs f_j plot.

The f_i vs f_j plot is a mapping process from the complex plane. The mapping functions of this process are the functions of the variable coefficients, namely $f_i = f_i(s)$. In this mapping process regions map into region, curves map into curves. An unbounded region on the S-plane maps an unbounded region unto the coefficient plane, while bounded regions map unto bounded regions. Therefore, the stable root regions discussed in the last Chapter map a stable region in the coefficient plane. The mapping process from S-plane to coefficient plane is unique, but the inverse (from coefficient plane to S-plane) mapping is not necessarily unique, it may have multi-values. By using this multi-value property of inverse mapping, more than one pair of complex roots can be evaluated from the coefficient plot. Figure 3-3 is a 4th order equation with f_1 and f_2 as variables. Fig. 3-3(a) shows the stable root-region and Fig. 3-3(b) shows the f_1 vs. f_2 plot. It can be seen from Fig. 3-3 that the unbounded stable root region on S-plane map an unbounded region unto $f_1 \sim f_2$ plane.

The $\zeta = 0.5$ lines on S-plane map a clockwise loop unto the plane. Every point on S-plane map a point on $f_1 \sim f_2$ plane. However, the inverse mapping is double-valued, such as points (2) and (3) on $f_1 \sim f_2$ plane map two points (2)' and two points (3)' on S-plane. Therefore, if sufficient curves have been drawn on $f_i \sim f_j$ plane, by using different contours (such as constant ζ and constant α lines), two pairs of complex-roots can be evaluated from the $f_i \sim f_j$ plane directly. For the case of a 4th order equation, all the roots can be seen on the $f_i \sim f_j$ plane for every choice of f_i and f_j. Real roots are not shown on the coefficient plot, but they can be evaluated easily from the reduced characteristic equation.



(a) Root-region

(b) $f_1 \sim f_2$ Plot

Fig. 3-3 Mapping of 4th Order Equation

Since for every region on the S-plane, there is a corresponding region on the coefficient plane, the required system root-region can be mapped on the coefficient plane if such a region exists. For example in Fig. 3-3, if the roots are required such that all roots have $\zeta = 0.5$ and time constant less than 1, then the root region of the equation has a corresponding region on the [illegible] plot which is the intersection region of the clockwise loops of $\alpha = 1$ and $\zeta = 0.5$. This region is shown by the shaded area. For other requirements, the region on the coefficient-plane which satisfies the requirement can be located by the same procedure.

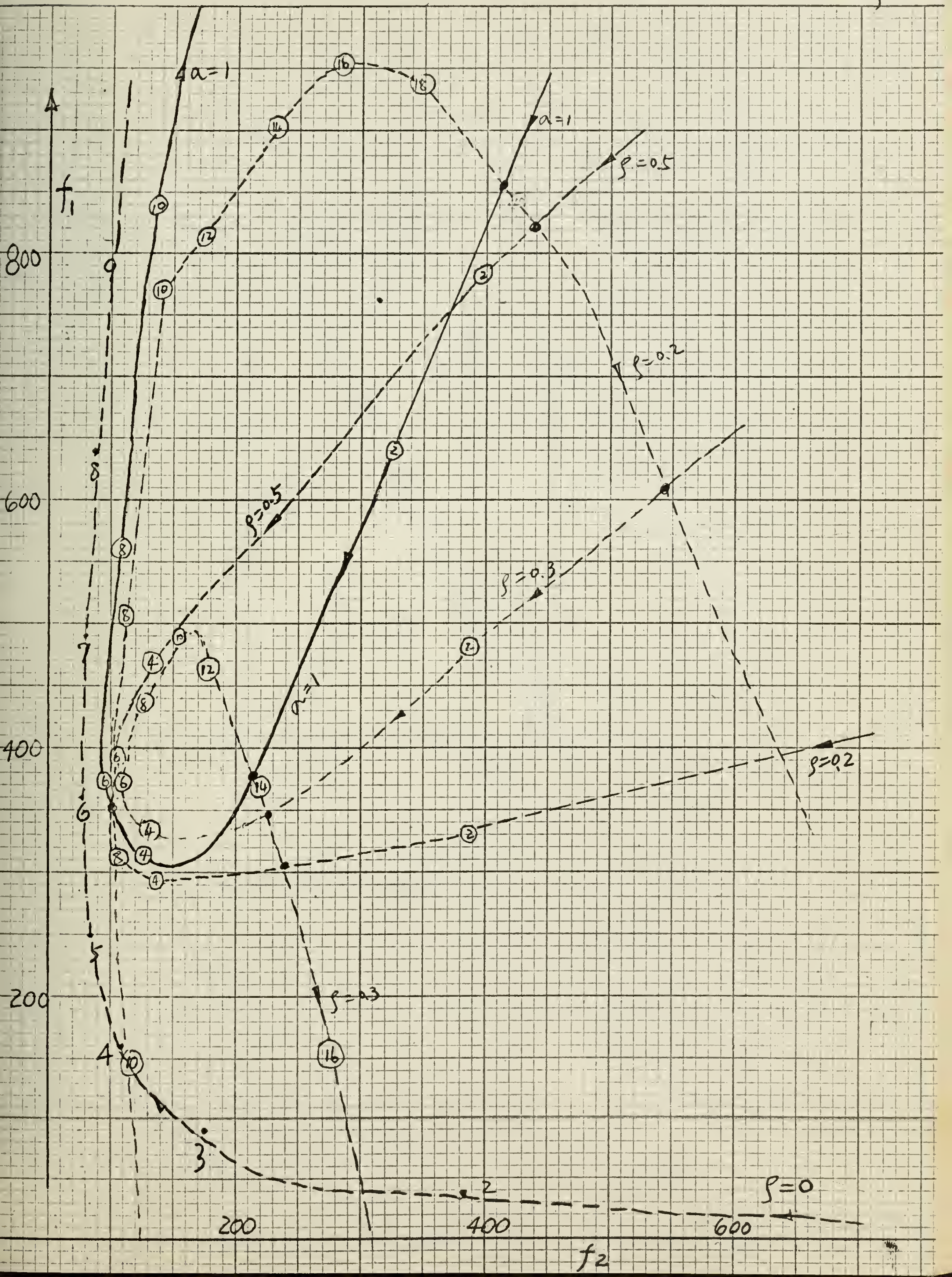
Figure 3-4 and Figure 3-5 are numerical examples with variables f_1 and f_2 for 4th order and 5th order equations respectively. In Figure 3-4 only one constant α (= 1) contour is drawn and in Figure 3-5, only one constant ζ (= 0.5) is drawn. From those figures, it can be seen that the constant ζ and constant α contour constitute curvilinear coordinates on the $f_1 \sim f_2$ plane. From the point of view of curvilinear coordinates, two pairs of complex-roots can be read for every point on the $f_1 - f_2$ plane.

3-7 Stability boundary on the coefficient-plane when the variable coefficients f_i are not in sequence:

As discussed in Section Three of Chapter Two, when the variable coefficients are not in sequence, the functions of the variable coefficients are undefined on the $j\omega$ axis. Consequently, the stability of the boundary line on the coefficient-plane cannot be obtained by mapping the $j\omega$ axis of the S-plane. However, the boundary on the S-plane is well defined as has been shown in Chapter Two, therefore, the boundary on the coefficient-plane can be obtained by mapping those boundary lines from the S-plane to the coefficient plane. This process can be carried out with point by point calculations or by approximating the boundary on the S-plane by a function.

The boundary in Fig. 2-12 has been calculated point by point and plotted in Fig. 3-6. In Fig. 3-6, a constant $\zeta = 0.1$, line is also mapped onto the $f_1 \sim f_3$ plane. The $\zeta = 0.1$ line shows a counterclockwise loop which implies that there exists a region in which all roots have $\zeta > 0.1$.

Fig. 3-4(a) plot of a 4th order equation. Solid line = constant α
Dash line = constant ρ



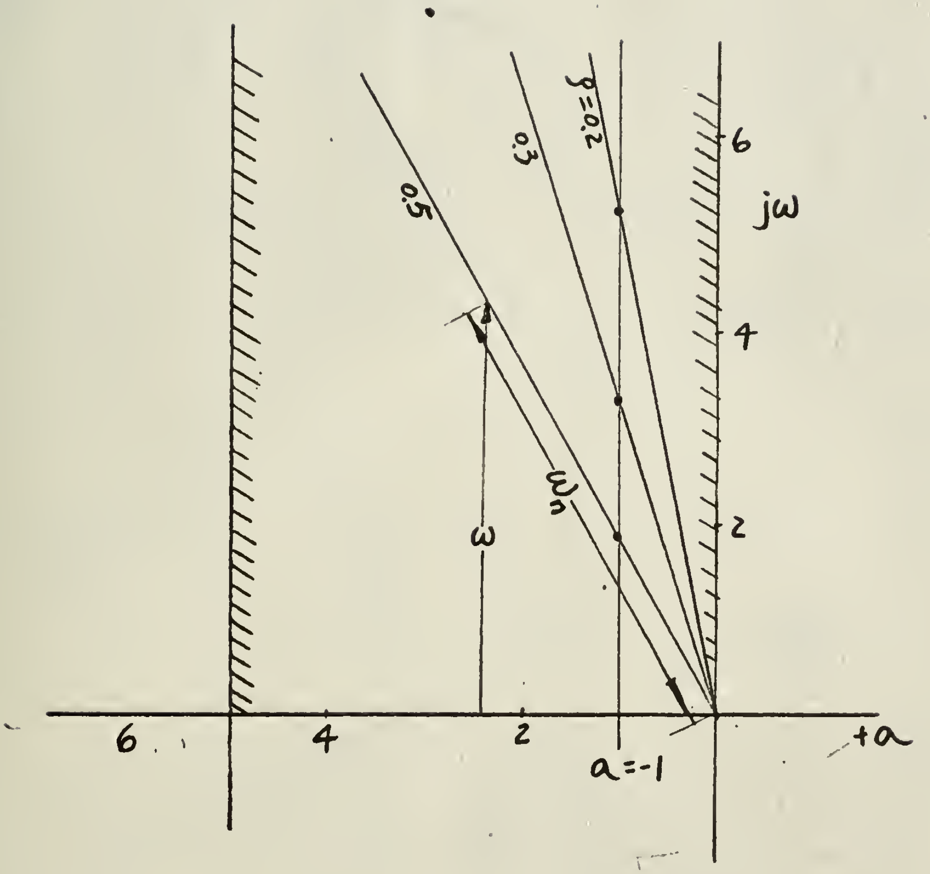


Fig. 3-4(b) Root-region

Fig. 5-5 Plot of a 5th equation solid line: constant
dash line : constant ρ

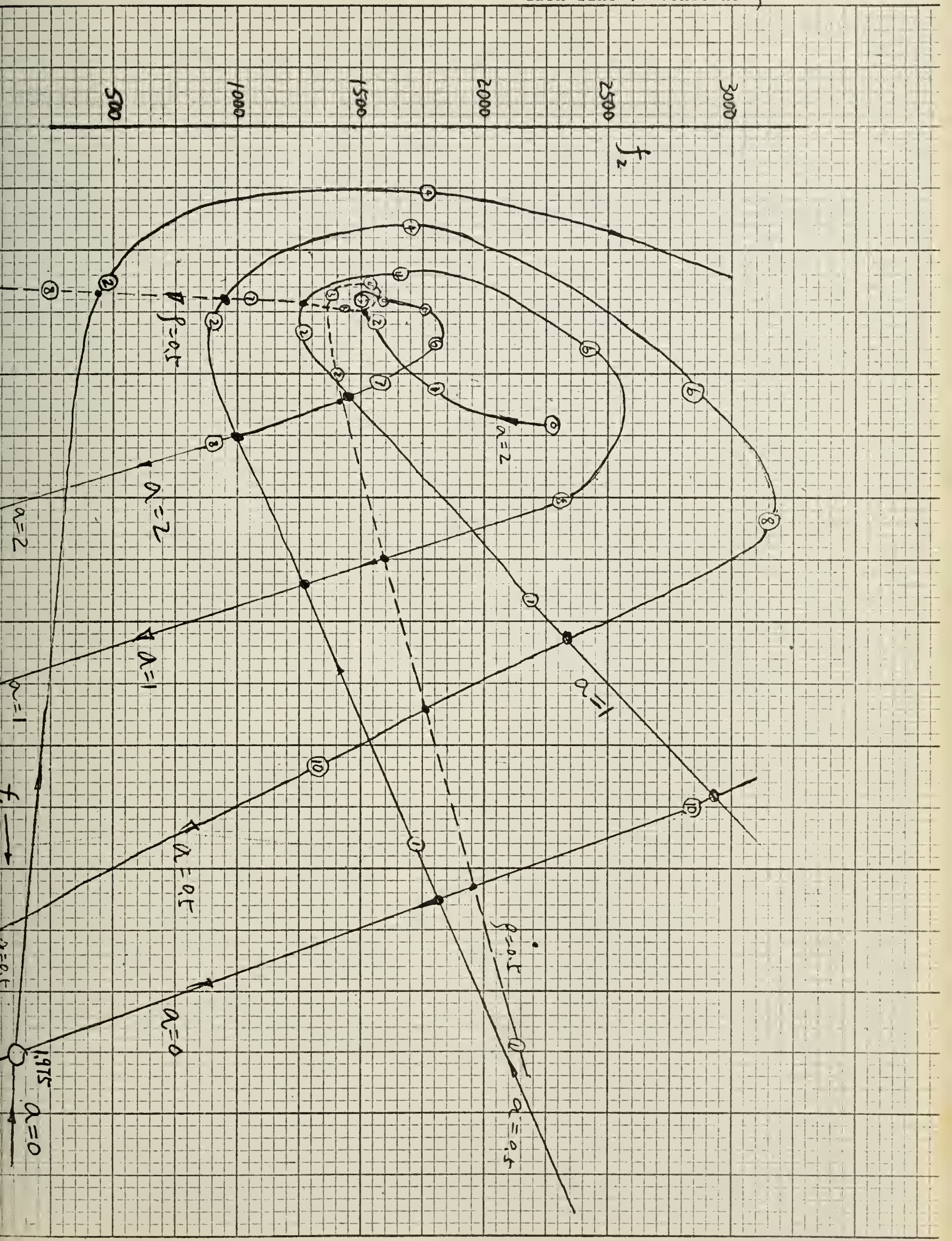
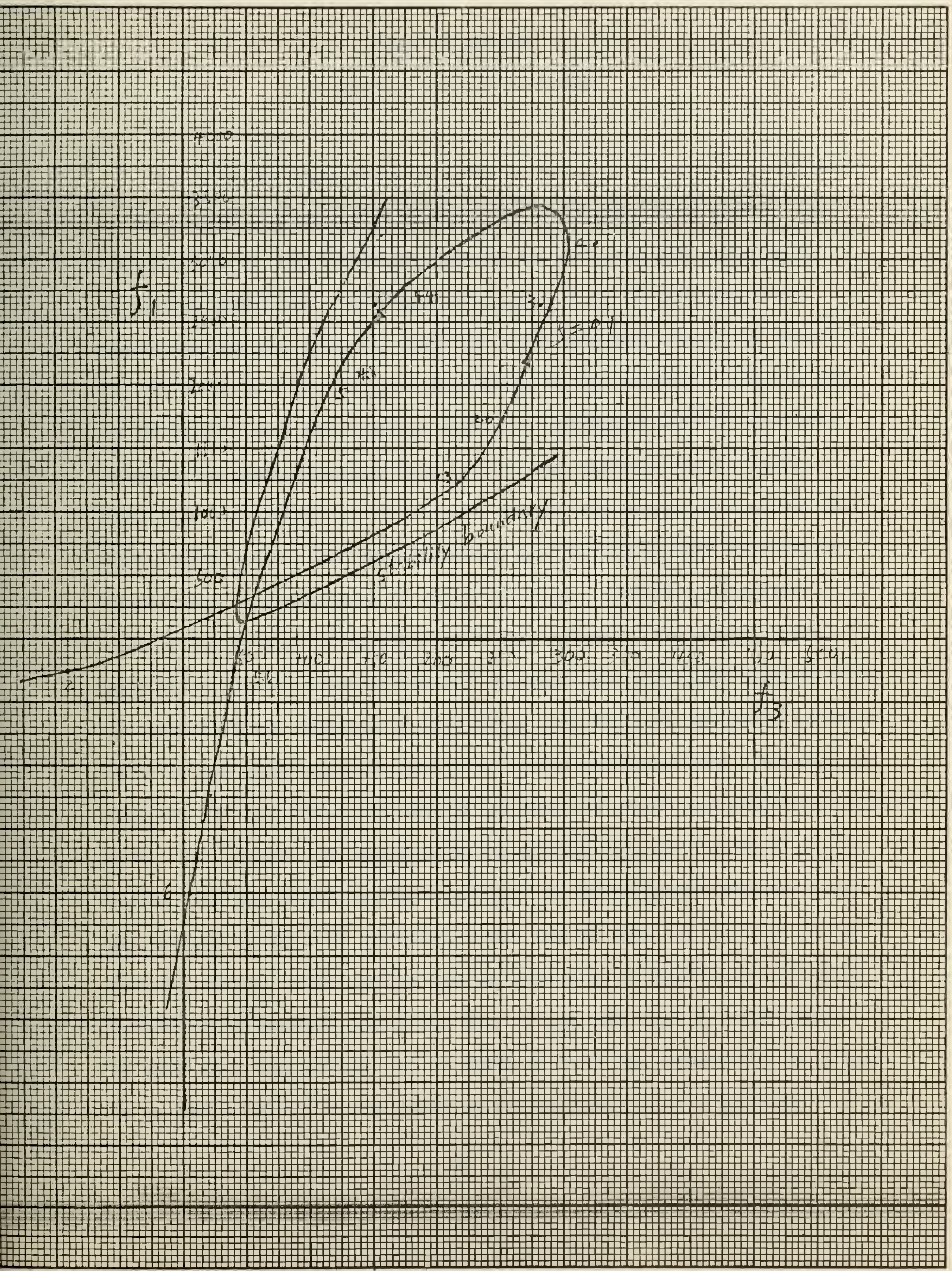


Fig. 3-6 Stability boundary on $f_1 \sim f_3$ plot.



PART II DESIGN

CHAPTER IV DESIGN PROCEDURE

4-1 Specifications.

The transient response of a control system is determined completely if the closed-loop pole-zero (root-zero) configuration is known. The contributions to the transient response by each pole and zero can be calculated precisely. If a pair of complex roots is dominant, then approximations can be made. Based on these dominant roots, the effect of all the zeros and all the secondary roots to the transient response can be visualized from the complete root-zero configuration.[6,1] With this relation the specifications from the desired response can be conveyed to root-zero configuration. As in this method, the design is based on root-zero configuration, all specifications must be interpreted in a set of appropriate configurations of roots and zeros. Frequency response is readily interpreted if dominant roots are assumed. The steady state specifications (stiffness and error constant) are contained implicitly in the equations and formulas to work with, then they are satisfied automatically if the final results satisfy all other specifications. However, to meet all the specifications may not be possible until some adjustments are made. For example, to design a system of high velocity constant (K_v) a dipole near the origin may be required. If the original dominant roots are defined so that all secondary roots are far away to the left of the dominant roots, it may result that no root-zero configurations for this definition of dominant roots are able to meet the high K_v requirement. In this case, the definitions of dominant roots may necessarily be adjusted so that a dipole to the right of the dominant roots is allowed.

In other words, a term of long tail of relatively small magnitude in the transient response is allowed. Usually, this long tail term in transient response is avoided. But in order to achieve high velocity error constant by a relatively simple network, this adjustment may be acceptable in consideration of all the factors in design.

4-2 Design Procedure -- General.

In the design of control systems, many methods and tools are available such as the root-locus method, frequency domain method, pole zero configurations, etc. Problems frequently rise about what procedures and patterns are more logical. In pure synthesis, the designer is able to take the specifications and proceeds logically in a step by step manner to a system which will meet these specifications. In these senses many of the design techniques are not true synthesis procedures. On the other hand, the design procedure proposed by Guillemin[1] which takes the form analogue to modern filter theory is a synthesis technique. In Guillemin's procedures the closed loop transfer function is first determined from the specifications and the compensation is determined subsequently. The design procedure by the analytic method developed in this paper is more or less close to Guillemin's procedure. Briefly, the procedures are as follows:

(1) The regions of closed loop pole-zero configurations are determined from the specifications.

(2) The type of compensator (cascade, feedback, pure derivative, or mixed network) is chosen from the consideration of specifications and other restrictions.

(3) Analytic expressions of all the roots and the parameters of compensation in terms of the arbitrary roots are obtained.

(4) Within the pole-zero region defined in (1), choose a set of

appropriate pole-zero configurations. Then the parameters of the compensator are calculated from the expressions in (3).

Procedure (1) is an interpretation or transition from the specifications to the closed loop pole-zero configurations. Procedure (2) depends upon the specifications, the nature of the plant and also on engineering judgment and experience. These two procedures are general to all methods, if the design is based on transient response or time domain. They are the prerequisites. Procedures (3) and (4) are concerned with the method itself. For this analytic method, they are explained in detail in the following sections.

4-3 Design Procedure -- Detail.

When the type of compensation has been chosen, the design procedure for this methid is as follows:

(1) Formulation of the Characteristic Equation.

The characteristic equation with the compensator is formulated in the way such that the coefficients of this equation are functions of the free parameters of the compensator. In other words, all parameters are independent from one another.

(2) Formulation of the Root-Coefficient Relations.

Set the coefficients in (1) equal to the corresponding root coefficient. If the order of the equation is "n" there are "n" relations. If there are "r" free parameters in the compensator, then "r" roots are arbitrary and the reduced characteristic equation is of the order (n-r) and has (n-r) coefficients. The root-coefficients relations are formulated in such a way that the coefficients are functions of the arbitrary roots and the coefficients of the reduced characteristic equation.

(3) Formulation of the Functions of the Dependent Variables.

In the "n" root coefficients relations formulated in Step 2, there are "n" dependent variables and r independent variables. The "n" dependent variables are the "r" parameters of the compensator and the (n-r) coefficients of the reduced characteristic equation. The independent variables are the "r" arbitrary chosen roots. Since there are "n" equations and "n" dependent variables, the dependent variables can be solved and expressed in functions of the arbitrary roots.

(4) Determination of Stability and Dominance.

From the reduced characteristic equation, determine the stability root-region and the dominant root-region. The dominant root region depends upon the degree and the definition of the dominance which is defined by the designer from the consideration of specifications.

(5) Evaluation of the Compensator Parameters.

If the desired dominant roots are within the dominant root-region which has been determined in step (4), then choose the dominant roots (usually independent variables) and substitute them into the formula of step (3), then the parameters of the compensator are evaluated.

(6) Realization of the Compensator Network.

If the parameters of the compensator evaluated in the last step are within the specified range and can be realized, then the design has been completed.

(7) Design Adjustment.

If the value of the compensator parameters are not in the specified range or the desired dominant roots are not in the dominant-root region, then it indicates either this type of compensation cannot meet the requirements or the definition of the dominant roots must be adjusted. When

other types of compensation have been chosen or the degree of dominance has been adjusted, the design procedure is just a repeat of the above steps.

The procedures described above are general. The first three steps are the partitioning process as discussed in Chapter I. For some specific cases, formula have been derived as shown in the subsequent sections, then the design is just a computation and is started from Step (4). The computation is best carried out by tabulating all previous data and by progressing computation. Once those data are tabulated, all information can be found from the table and if some changes of the type of the compensation are made, most of the data can be utilized.

4-4 Free Parameters of the Compensator.

In step (2) of the last section, a terminology "Free Parameters" was used. It may need further explanation. Consider a cascade single section R-C compensator. Let the transfer function be

$$G_c = \frac{K_c(s+z)}{s+p}$$

The apparent parameters are K_c, Z and p. Assume the system is type I and the velocity error constant K_v is specified. If the forward gain K of the uncompensated system was determined by the K_v requirement and in the partitioning processes, K was treated as a given numerical constant then K_c is not arbitrary in order to meet the requirement of K_v. In other words K_c must satisfy the following relation:

$$K_c = P/Z$$

Thus the three apparent compensator parameters K_c, p, and Z are related by the requirement of K_v. Only two of them are independent variables, the third one is dependent. The independent variables of the compensator in this sense are given the terminology "free parameters".

For other cases, the same treatment can be applied to the compensator parameters. In the above example, if the forward gain K was also treated as an adjustable parameter, then all the three apparent parameters are independent because $K_c K$ is also independent. However, the result of the partitioning processes are different for the two cases. But the design can be carried out by either case. In the former case (K is constant) there are two arbitrary roots, while in the latter case (K is adjustable) there are three arbitrary roots. The expressions of the parameters are different for different cases, but the design can be carried out in both cases.

5-1 Tachometer Feedback - Second Order.

For compensating lightly damped instrument servos, tachometer feedback is commonly used. For second order cases, as all information can be determined precisely, only the formulas are presented.

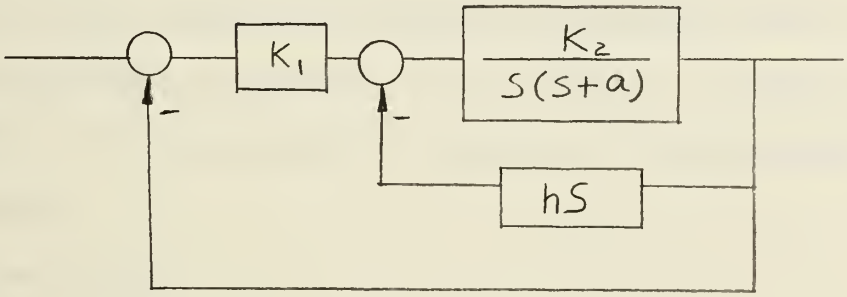


Fig. 5-1 Second Order System with Tachometer Feedback

The block diagram of a Type 1 second order system is shown in Fig. 5-1. The separation of the forward gain by K_1 and K_2 is for the purpose of proper adjustment of h. Assume the specifications are given K_1K_2 and ζ . The design procedures are as follows:

(1) Formulation of the Characteristic Equation

The Characteristic Equation is

$$S^2 + (a + K_2 h)S + K_1 K_2 = 0 \tag{5-1}$$

(2) Formulation of the Root-Coefficient Relation.

Since there is only one parameter of compensation, there is only one independent variable. The independent variable is not necessarily the arbitrary root itself, it can be a function of the roots. In the case of one arbitrary root and that root has to be complex, the independent variable can be chosen as either ζ or ω_n of the pair of complex conjugate roots. For second order case, the only choice is ζ ,

because ω_n is fixed by the gain. By doing this the root-coefficient relations are:

$$a + K_2 h = 2\zeta\omega_n \tag{5-2-1}$$

$$K_1 K_2 = \omega_n^2 \tag{5-2-2}$$

(3) Formulation of the Functions of the Dependent Variables.

Since the root of the reduced characteristic equation is the dominant root itself, there are no stability and dominant region problems. Here the dependent variables are ω_n and h(ζ is chosen as independent variable). Their expressions as functions of the independent variable are as follows:

From (5-2-2):

$$\omega_n = \sqrt{K_1 K_2} \tag{5-3-1}$$

From (5-2-1): $K_2 h = 2\zeta\omega_n - a = 2\zeta\sqrt{K_1K_2} - a$

$$h = \frac{2\zeta\sqrt{K_1K_2} - a}{K_2} \tag{5-3-2}$$

For the given value of ζ , h is computed.

5-2 Third Order: Tachometer Feedback.

The block diagram of a third order type one system with tachometer feedback all around the plant is shown in Fig. 5-2.

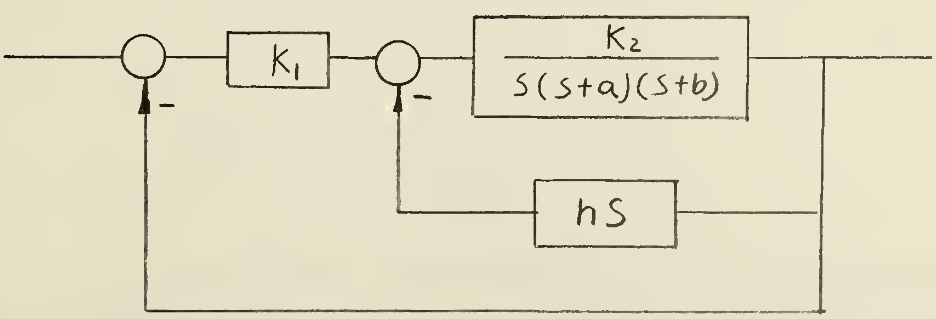


Fig. 5-2 Third Order System with Tachometer Feedback

The characteristic equation of the uncompensated system is

$$S^3 + B_2 S^2 + B_1 S + B_o = 0 \tag{5-3}$$

where $B_2 = a + b$, $B_1 = ab$, $B_o = K_1 K_2$.

Assume the forward gain K and the damping ratio ζ based on the dominant roots are given. The design procedures are:

(1) Formulation of the characteristic equation: It is easily formulated as follows:

$$S^3 + B_2 S^2 + (B_1 + K_2 h) S + B_o = 0 \tag{5-4}$$

(2) Formulation of the root-coefficient relation: Since there is only one control parameter h, only one root is arbitrary. This root can be chosen either the real root or ζ of the complex roots. Assume the real root is chosen and designated as $-S_1$ (S_1 is positive on the negative real axis). Then S_1 is the independent variable. The reduced characteristic equation is of order 2 and therefore has the form

$$S^2 + C_1 S + C_o = 0$$

Since the roots of this equation are actually the dominant roots, it is expressed by ζ and ω_n for convenience. Here ζ and ω_n are not independent variables but dependent variables. By the choice of the above definition, the root-coefficients relations are:

$$B_2 = 2\zeta\omega_n + S_1 \tag{5-5-1}$$

$$B_1 + K_2 h = \omega_n^2 + 2\zeta\omega_n S_1 \tag{5-5-2}$$

$$B_o = \omega_n^2 S_1 \tag{5-5-3}$$

(3) Determination of the functions of the dependent variables. Here the three dependent variables are ζ, ω_n and h. In the actual computation, only one dependent variable must be found explicitly

as a function of the independent variable, the others can be computed by using the implicitly functions of formulas (5-5). But for the purpose of illustration, they are solved and shown in equations (5-6)

$$\zeta = (B_2 - S_1)\sqrt{S_1} / 2\sqrt{B_0} \tag{5-6-1}$$

$$\omega_n = \sqrt{B_0 / S_1} \tag{5-6-2}$$

$$K_2 h = B_0/S_1 + B_2 S_1 - S_1^2 - B_1 \tag{5-6-3}$$

Note the three variables ζ , ω_n and h are expressed as functions of the independent variable.

(4) Determination of stability and dominant root region:

In this one control parameter case, the stability and dominant root regions are lines. These lines are the same as the root locus, by treating h as variable and K as a fixed number. The only difference between them is the independent variable. In the root-locus method, the independent variable is h while in this method it is the arbitrary root S_1. A numerical example is shown in Fig. 5-4. The degree of dominance can be visualized easily from those plots.

For this particular scheme, it indicates a max. damping. This max. damping can be found from equation (5-6-1) analytically because ζ is expressed explicitly as a function of S_1. Differentiate ζ with respect to S_1 and set it to zero, one obtains:

$$S_{1m} = B_2/3 \tag{5-7}$$

Substitute into equation (5-6-1), obtain

$$\zeta_{max}^2 = B_2^2 / 27 B_0 \tag{5-8}$$

Substitute this value of S_1 into ζ and ω_n , the real part of the complex root becomes

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$$\zeta \omega_n = B_2/3 = S_{1m} \tag{5-9}$$

Equation (4-7) shows that if the tachometer loop gain K_2h is so adjusted such that the real root is ($-\frac{B_2}{3}$) then the complex roots have max. damping for the given value of B_o(= K) and this max. damping is shown in equation (5-8). Equation (5-9) shows that for max. damping the real part of the complex roots is equal to the real root. Fig. 5-3 shows this root configuration.

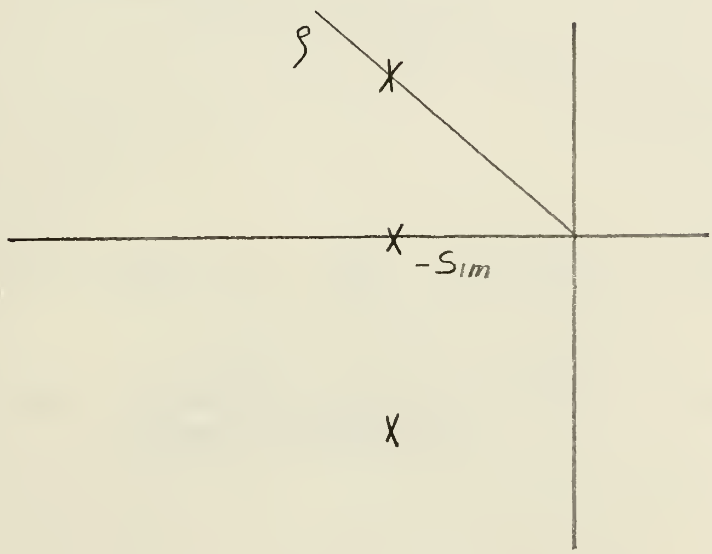


Fig. 5-3 Root Configuration for max. ζ of 3rd Order System with Tachometer Feedback.

For this root configuration, the tachometer loop gain and ω_n are obtained from equations (5-6-2) and (5-6-3) as follows:

$$K_2h = \frac{3B_o}{B_2} + \frac{2}{9}B_2^2 - B_1 \tag{5-10}$$

$$\omega_n = \sqrt{\frac{3B_o}{B_2}} \tag{5-11}$$

Equation (5-8) gives a quick look of the max. possible damping for given K or the max. K for a given ζ in a third order system with tachometer

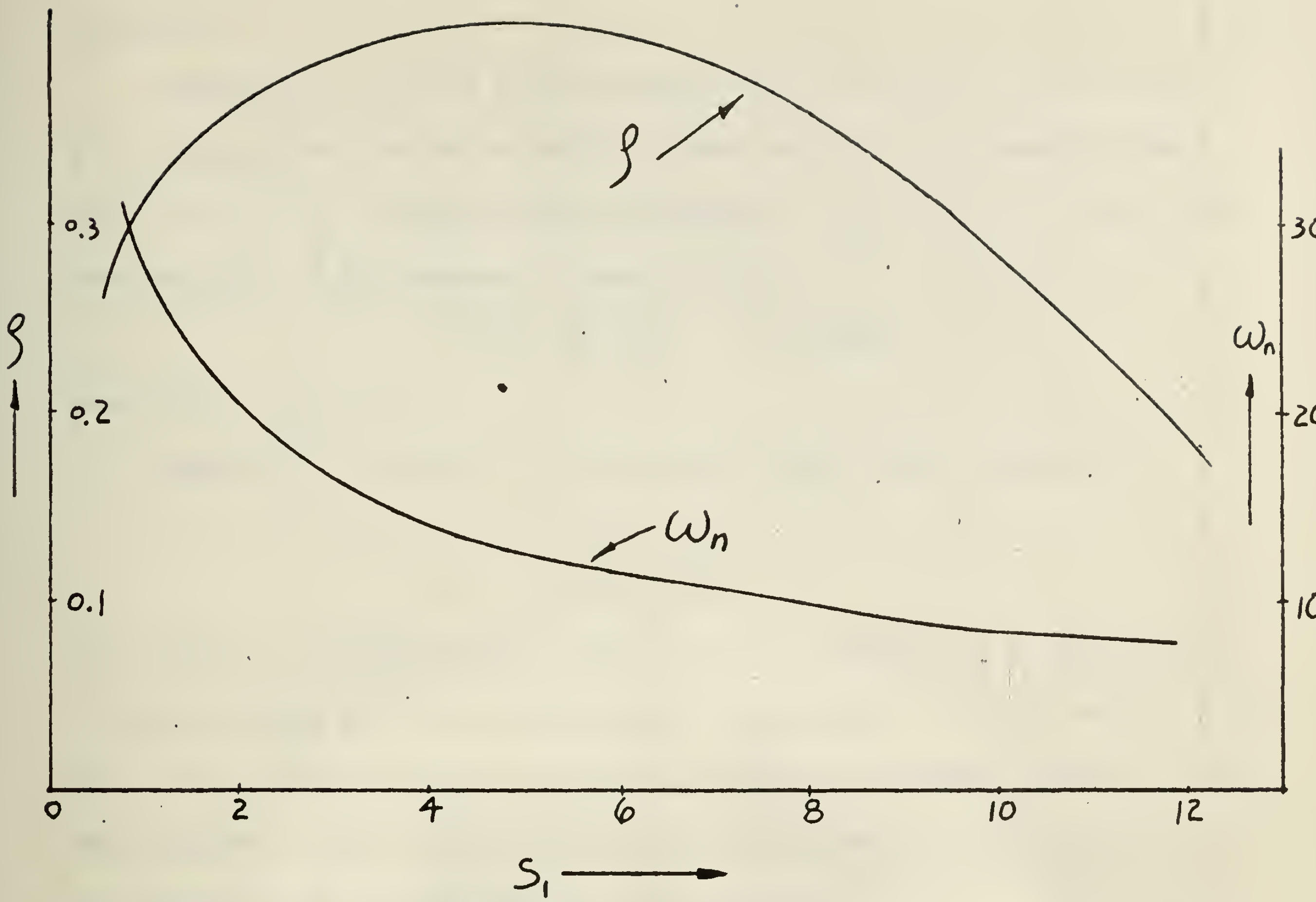


Fig. 5-4(a) Plot of Equations (5-6-1) and (5-6-2) for numerical values given in example 5-1.

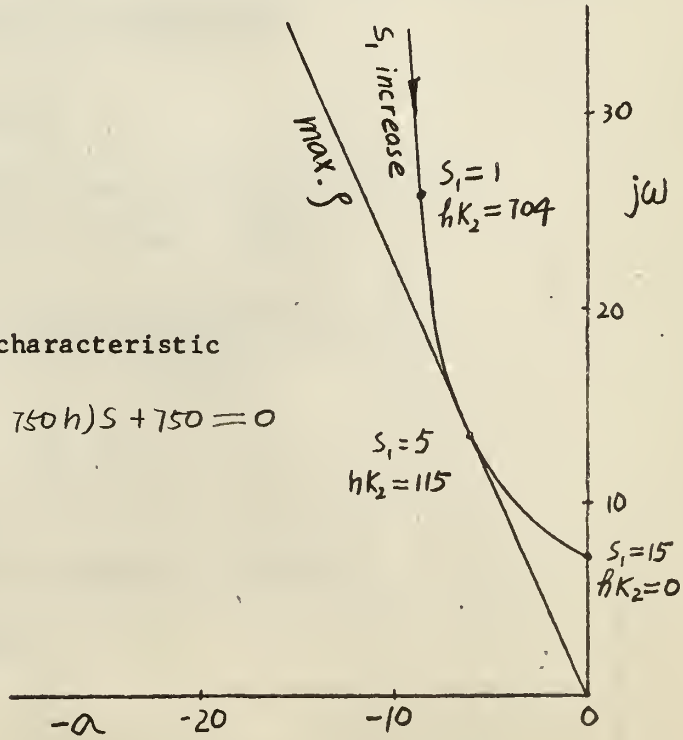


Fig. 5-4(b) Root locus of the characteristic Equation

$$S^3+15S^2+(50+750h)S+750=0$$

feedback.

Example 5-1: For the block diagram shown in Fig. 5-2, if $b = 10$, $a = 5$, $K_1K_2 = 750$; the uncompensated system has a pair of complex roots on $j\omega$ axis. For this pole-zero configuration: $B_2 = 15$, $B_1 = 50$, $B_o = 750$. From equation (5-8), the max. possible damping

$$\zeta = \sqrt{\frac{15^3}{27 \times 750}} = 0.408$$

From (5-7) $S_{1m} = 5$.

Assume this value of ζ is taken, then from (5-6-2) and (5-6-3)

$$\omega_n = 12.3$$

$$K_2 h = 18.5$$

For other root configurations, ζ and ω_n as a function of S_1 are plotted in Fig. (5-4)(a) and the root-locus for h as variable is shown in Fig. (5-4)(b). If the dominant root is defined so that the secondary root has greater real part than that of the dominant roots, then the portion of the root locus when $S_1 \geq 5$ is the dominant root region.

5-3 Third Order: Tachometer Feedback to Interior Node.

Consider Fig. 5-5 in which point "X" is assumed to be available to insert a summer. The analytic design procedures are:

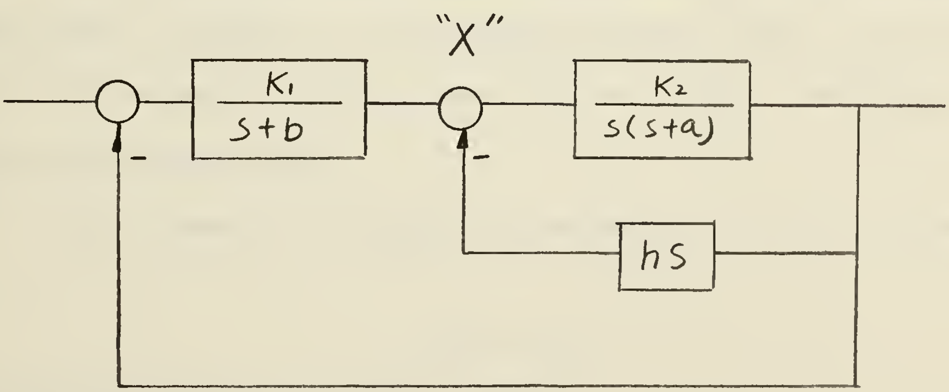


Fig. 5-5 Tachometer Feedback: Minor Loop

(1) The characteristic equation is:

$$s^3 + (B_2 + K_2 h)s^2 + (B_1 + bK_2 h)s + B_o = 0 \qquad (5\text{-}12)$$

where $B_2 = a+b$, $B_1 = ab$, $B_o = K_1 K_2$

(2) Root-coefficient relationship. For the same definition of dependent and independent variables as described in the last section, the relations are

$$B_2 + K_2 h = 2\zeta\omega_n + S_1 \qquad (5\text{-}13\text{-}1)$$

$$B_1 + bK_2 h = \omega_n^2 + 2\zeta\omega_n S_1 \qquad (5\text{-}13\text{-}2)$$

$$B_o = \omega_n^2 S_1 \qquad (5\text{-}13\text{-}3)$$

(3) Function of dependent variables (ζ , ω_n, $K_2 h$).

From equation (5-13-1) and (5-13-2), eliminate K_2h

$$\zeta = (B_2 b - B_1 + \omega_n^2 - bS_1) / 2\omega_n (b - S_1)$$

Substitute ω_n from (5-13-3) into the above expression obtain

$$\zeta = \frac{B_2 b - B_1 + \frac{B_o}{S_1} - bS_1}{2(b - S_1)\sqrt{B_o/S_1}} \qquad (5\text{-}14)$$

The other two variables can be obtained in the same way but are not necessary.

(4) Stability and dominant root region: For the numerical values given in Example 5-1, the dominant roots ζ and ω_n are plotted in Fig. 5-6 as functions of S_1 from equation (5-14). The degree of dominance can be seen easily.

(5) Take $\zeta = 0.5$ which corresponding to $S_1 = 20$. From (5-13-3)

$$\omega_n = 6.14$$

From either (5-13-1) or (5-13-2), obtain

$$K_2 h = 11.14$$

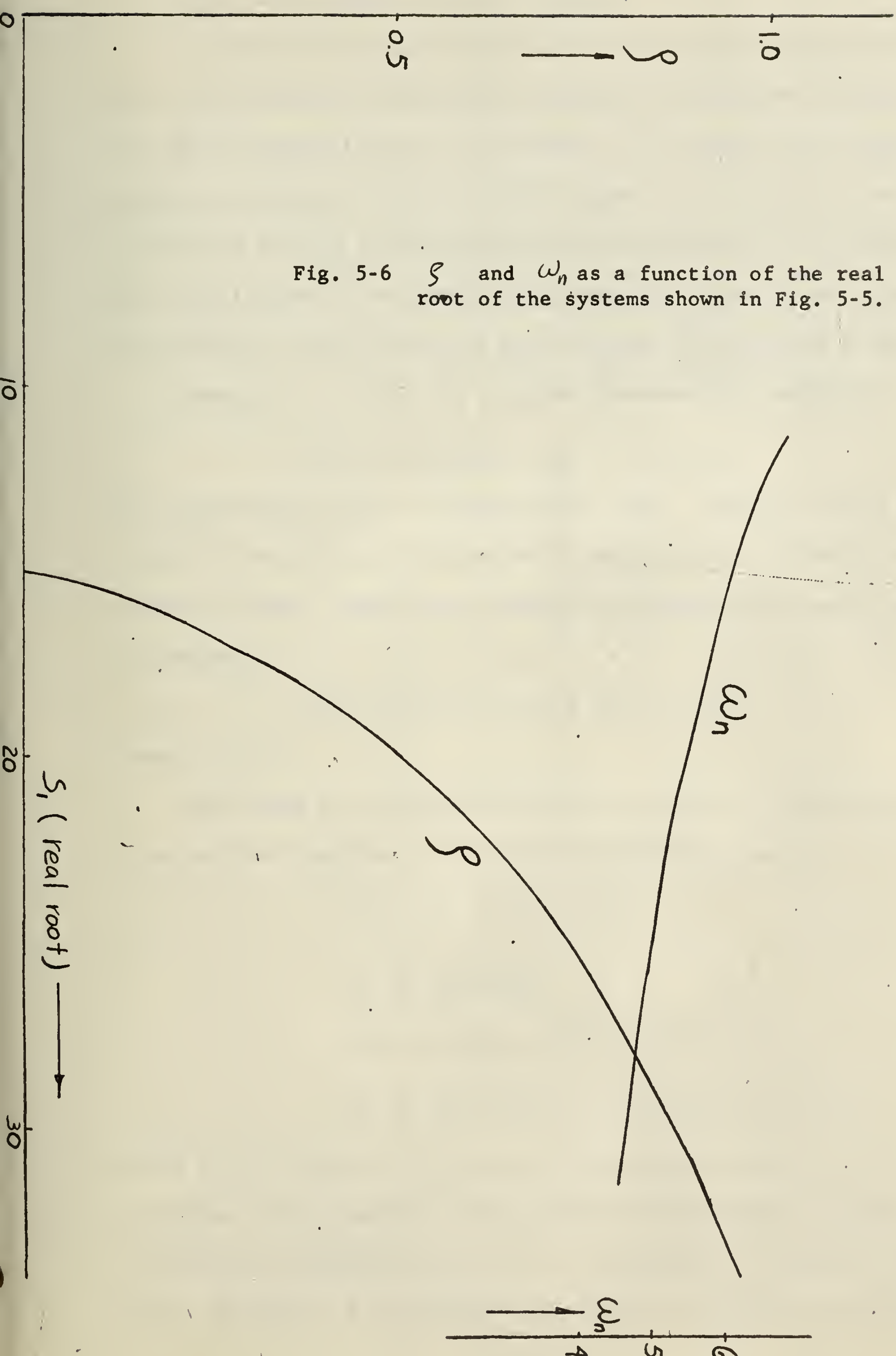


Fig. 5-6 ζ and ω_n as a function of the real root of the systems shown in Fig. 5-5.

5-4 First and Second Derivative Feedback.

In case of first and second derivative feedback around the plant for a given value of forward gain, only the coefficients of the first and second derivative term are variable. The design may be carried out using two variables. But for more freedom it is better to take the constant term also as a variable then three arbitrary roots are available. The coefficients of the reduced characteristic equation and functions of the variable coefficients have been derived and tabulated in Table 1-6.

Example 5-2. Given the original characteristic equation,

$$S^4 + 10S^3 + 31S^2 + 30S + K = 0 \tag{5-15}$$

The forward gain K must be greater than 1000. The gain at the stability limit for the uncompensated system is 84, first and second derivative feedback are chosen. When three variable coefficients are used, equation (5-15) becomes

$$S^4 + 10S^3 + f_2 S^2 + f_1 S + f_0 = 0 \tag{5-16}$$

where $f_0 = K$

From Table 1-6, the coefficient of the reduced characteristic equation and the functions of the variable coefficients are:

$$C_0 = B_3 - (S_1 + S_2 + S_3) \tag{5-17-1}$$

$$f_0 = C_0 S_1 S_2 S_3 \tag{5-17-2}$$

$$f_1 = C_0 (S_1 S_2 + S_2 S_3 + S_3 S_1) \tag{5-17-3}$$

$$f_2 = C_0 (S_1 + S_2 + S_3) + (S_1 S_2 + S_2 S_3 + S_3 S_1) \tag{5-17-4}$$

where $(-S_1)$, $(-S_2)$ and $(-S_3)$ are the arbitrary roots and $C_0 = S_4$. Here the order of the equation is four, all the roots may be complex. There is only one constrained root so it is convenient to replace C_0 by S_4 itself. The order of the reduced characteristic equation is one; equation

(5-17-1) alone determines all the roots. Assume $(-S_1)$ and $(-S_2)$ are the dominant roots, then (5-17-1) becomes

$$10 = 2\zeta\omega_n + S_3 + S_4 \tag{5-18-1}$$

$$f_o = \omega_n^2 S_3 S_4 \tag{5-18-2}$$

There are no zeros in this system, all secondary roots ($-S_3$ and $-S_4$) are required to be to the left of the dominant roots. Assume $(-S_3)$ and $(-S_4)$ are real roots, then for the definition of dominance, the max. K occurs when $S_3 \doteq S_4 = \zeta\omega_n$. From (5-18-1) for this value of the secondary roots, the conditions are:

$$\zeta\omega_n = 2.5 \tag{5-19-1}$$

$$\max.\ K = \zeta^2\omega_n^2 \tag{5-19-2}$$

For $\zeta = 0.5$, the max. K = 156. This is far below the required K, then the secondary roots must be complex. Assume ζ' and ω_n' denote the second roots, then equations (5-18) become

$$\zeta\omega + \zeta'\omega_n' = 5 \tag{5-20-1}$$

$$K = \omega_n^2\omega_n'^2 \tag{5-20-2}$$

Notice in equation (5-20-1) there are three arbitrary variables and this equation alone determines all the roots. Assume the dominance is defined as $\zeta'\omega_n' > \zeta\omega_n$ then from (5-20-1), the dominant root region is $\zeta\omega < 2.5$, which is shown in Fig. 5-7a. Within this region choose $\zeta = 0.5$, $\omega_n = 2$, then from (5-20-1) the condition for secondary roots is $\zeta'\omega_n' = 4$

Substitute into (5-20-2), obtain $K = 64/\zeta'^2$

For K = 1000, obtain $\zeta' = 0.253$. For this choice of the dominant roots

f_1 and f_2 are readily calculated from (5-17-3) and (5-17-4) as follows

$$f_1 = 532$$

$$f_2 = 270$$

The root configuration is shown in Fig. 5-7b. For another choice of the dominant roots, the computation can be carried out in just the same way.

Example 5-3: Given the original characteristic equation

$$s^5 + 16s^4 + 128s^3 + 520s^2 + 1300s + K = 0$$

The forward gain K must be greater than 2000. For this value of K the system is badly damped, first and second derivative feedback are chosen. By using

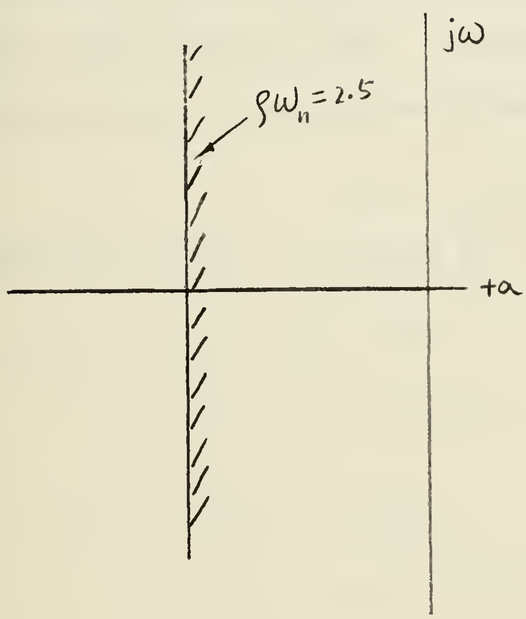


Fig. 5-7a
Dominant Root Region of
Example 5-2

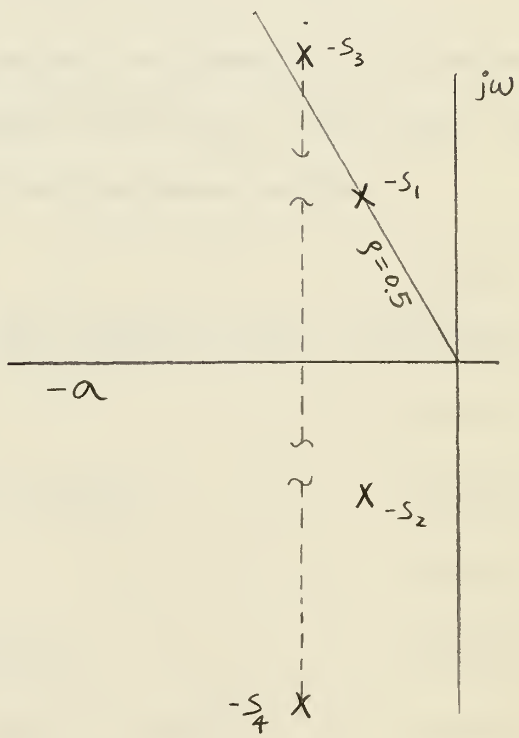


Fig. 5-7b
Root -configuration of Example 5-2
for $\zeta = 0.5, \omega_n = 2$

three variable coefficients, the working formulas from Table 1-6 are as follows:

$$C_1 = B_4 - (S_1 + S_2 + S_3) \tag{5-21-1}$$

$$C_0 = B_3 - C_1(S_1 + S_2 + S_3) - (S_1S_2 + S_2S_3 + S_3S_1) \tag{5-21-2}$$

$$f_0 = S_1S_2S_3C_0 \tag{5-21-3}$$

$$f_1 = C_0(S_1S_2 + S_2S_3 + S_3S_1) + C_1S_1S_2S_3 \tag{5-21-4}$$

$$f_2 = C_0(S_1 + S_2 + S_3) + C_1(S_1S_2 + S_3S_2 + S_1S_3) + S_1S_2S_3 \tag{5-21-5}$$

Assume $(-S_1)$ and $(-S_2)$ are the dominant roots. The order of the equation is five, one root must be real. Let this real root be $(-S_3)$. Here three roots are arbitrary. S_3 may assume any value. For dominance, assume

$$S_3 = 2\zeta\omega_n \tag{5-22}$$

Substitute (5-22), $B_4 = 16$ and $B_3 = 128$ into equation (5-21), one obtains:

$$C_1 = 16 - 4\zeta\omega_n \tag{5-23-1}$$

$$C_0 = 128 - 4\zeta\omega_n C_1 - (1 + 4\zeta^2)\omega_n^2 \tag{5-23-2}$$

$$f_0 = 2\zeta\omega_n^3 C_0 \tag{5-23-3}$$

$$f_1 = C_0\omega_n^2(1 + 4\zeta^2) + 2\zeta\omega_n^3 C_1 \tag{5-23-4}$$

$$f_2 = 4\zeta\omega_n C_0 + (1 + \zeta^2)\omega_n^2 C_1 + 2\zeta\omega_n^3 \tag{5-23-5}$$

Notice (5-23-1) and (5-23-2) determine all the roots, (5-23-3) determines the forward given K while (5-23-4) and (5-23-5) determine the derivative gain. Since there are no zeros in the system, the secondary roots are required to be to the left of the dominant roots. Let $a = \zeta\omega_n$, the transformed coefficients from Table 2-4 are as follows:

$$D_1 = C_1 - 2\zeta\omega_n \tag{5-24-1}$$

$$D_0 = C_0 - \zeta\omega_n C_1 + \zeta^2\omega_n^2 \tag{5-24-2}$$

The stability boundary lines are determined by $C_1 > 0$ and $C_0 > 0$.

The dominant boundary lines are determined by $D_1 > 0$ and $D_0 > 0$.

They are shown in Fig. 5-8a. For ζ = 0.5, the results of computation of C_0, C_1, f_0, D_0 and D_1 are tabulated in Table 5-1. Examine Table 5-1, when ω_n is between 4 and 5 the gain K is greater than the specified value 2000. This range of ω_n is within the dominant region since both D_1 and D_0 are positive. The root-configuration for ω_n = 4, ζ = 0.5, is shown in Fig. 5-8b. For this choice of the dominant roots, the variable coefficients are computed readily from (5-23-3), (5-23-4) and (5-23-5).

$$f_0 = K = 2040$$

$$f_1 = 1534 \text{ (original 1300)}$$

$$f_2 = 576 \text{ (original 520)}$$

For other values of S_3 and other choices of the dominant root the computation is the same.

ω_n	C_1	C_0	f_0	D_1	D_0
2	12	72	576	+	+
3	10	50	1350	+	+
4	8	32	2040	+	+
5	6	18	2250	+	+
6	4	8	1728	-	+
7	2	2	686		+
8	0	0	0		

Table 5-1 Example 5-3 for ζ = 0.5

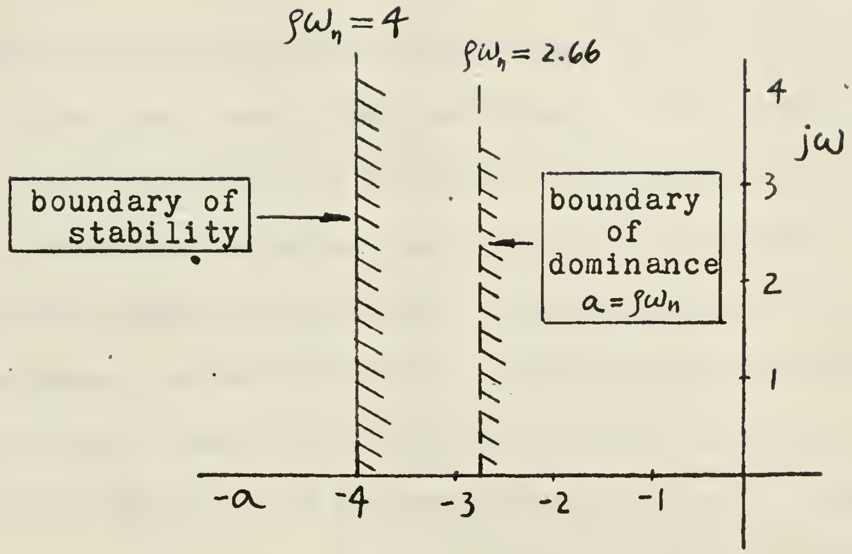


Fig. 5-8a Stability and Dominant Root Regions of Example 5-3

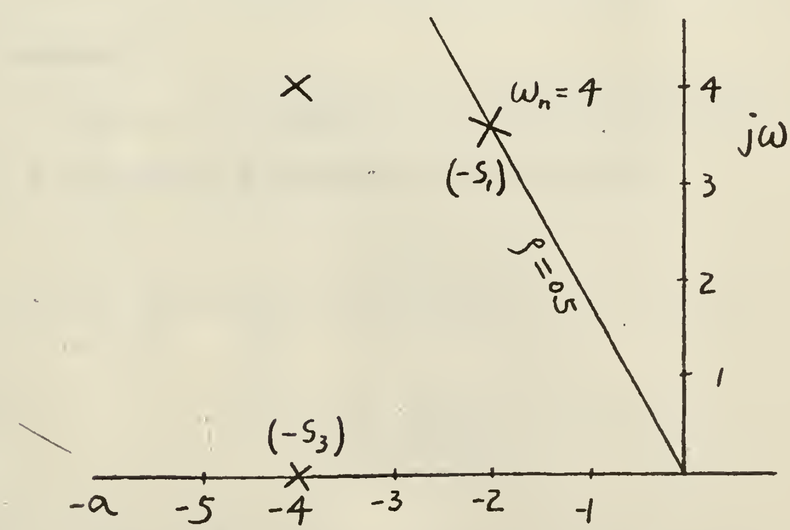


Fig. 5-8b Root Configuration of Example 5-3 for $\zeta = 0.5$ $\omega_n = 4$.

CHAPTER VI

FEEDBACK COMPENSATOR DESIGN

6-1 Introduction.

The compensation by pure derivative feedback discussed in the last chapter usually adjusts only some of the coefficients of the characteristic equation. The effectiveness of this type of compensation is less than that if the compensator parameters enter all the coefficients. This has been shown by the reduced characteristic equation in Chapter II. Moreover, the tachometer feedback reduced K_v and acceleration feedback reduced K_a and so on; and a system can seldom be stabilized without adjusting the lower order coefficients of the characteristic equation. Therefore, if the compensator is chosen in the feedback path, the compensator often consists of a tachometer followed by a network. The tachometer effectively changes the coefficient of the first derivative term, while the poles and zeros of the network enter all the coefficients in the way of the coherent nature of algebraic equations. This chapter is a discussion about the design of such a compensator by the analytic method.

6-2 Second Order Compensator.

Consider Fig. 6-1 which is a second order system with feedback compensator consisting of a tachometer followed by a lead network.

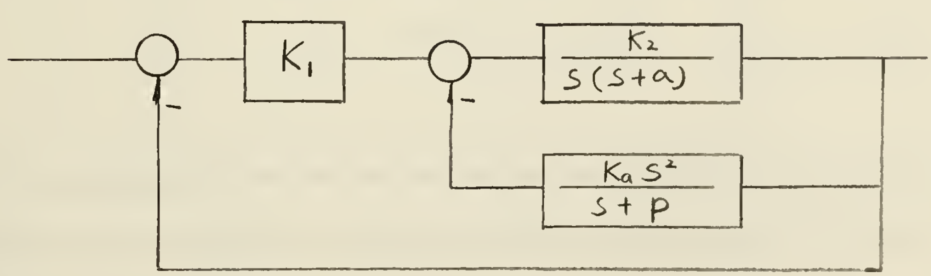


Fig. 6-1 Second Order System with Feedback Compensator.

The characteristic equation of the compensated system is

$$S^3 + (B_1 + p + K_2 k_a)S^2 + (B_0 + B_1 p)S + B_0 p = 0 \qquad (6\text{-}1)$$

where $B_1 = a$, $B_o = K_1 K_2$. In this characteristic equation, there are two variables in the coefficients, therefore two roots are arbitrary if K_a and p have no restrictions. The reduced characteristic equation is of order one and has the form of $S + C_o = 0$. The root coefficient relations are:

$$B_1 + p + K_2 k_a = 2\zeta\omega_n + C_o \qquad (6\text{-}2\text{-}1)$$

$$B_o + B_1 p = \omega_n^2 + 2\zeta\omega_n C_o \qquad (6\text{-}2\text{-}2)$$

$$B_o p = C_o \omega_n^2 \qquad (6\text{-}2\text{-}3)$$

In equation (6-2), the dependent variables are p, k_a and C_o; the independent variables are ζ and ω_n which are assumed to define the dominant roots. C_o determines the stability and dominant root region, while p and k_a must assume values according to the dominant roots. Since equations (6-2) are linear functions of the dependent variables, it is convenient to write in the matrix form as follows:

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & B_1 & -2\zeta\omega_n \\ 1 & B_o & -\omega_n^2 \end{bmatrix} \begin{bmatrix} K_2 k_a \\ p \\ C_o \end{bmatrix} = \begin{bmatrix} 2\zeta\omega_n - B_1 \\ \omega_n^2 - B_o \\ 0 \end{bmatrix} \qquad (6\text{-}3)$$

The three dependent variables can be solved readily, but only C_o must be expressed as an explicit function of the independent variables. By Cramer's rule, one obtains

$$C_o = \frac{B_o - \omega_n^2}{2\zeta\omega_n - \frac{B_1}{B_o}\omega_n^2} \qquad (6\text{-}4)$$

C_o in this case is the actual real root. The stability and dominant root region are determined by equation (6-4) by varying ζ and ω_n. For design purposes, only those values of ζ and ω_n which are within the specifications are computed. When ζ and ω_n have been chosen, K_2ka and ρ are calculated and the design is completed:

Example 6-1: In the block diagram Fig. 6-1, $K_1K_2 = 10^5$, $a = 23$. Therefore $B_o = 10^5$ and $B_1 = 23$. From equation (6-4):

$$C_o = \frac{(10^5 - \omega_n^2)\,10^5}{2\times10^5 \zeta\omega_n - 23\omega_n^2}$$

The stability criterion is $C_o > 0$ i.e,

$$\frac{10^5 - \omega_n^2}{2\times10^5\zeta\omega_n - 23\omega_n^2} > 0$$

or

$$10^5 - \omega_n^2 > 0 \quad \text{and} \quad 2\times10^5\zeta\omega_n - 23\omega_n^2 > 0$$

These two inequalities define the stability region and is shown in Fig. 6-2. If the dominant region is defined by $C_o > 2\zeta\omega_n$ then the criterion for this definition is

$$\frac{10^5 - \omega_n^2}{2\times10^5\zeta\omega_n - 23\omega_n^2} > 2\zeta\omega_n$$

This region is also shown in Fig. 6-2. For $\zeta = 0.5$, the range of ω_n is $0 < \omega_n < 220$. The values of C_o, ρ and K_2ka for the dominant roots in this region ($\zeta = 0.5$) are computed as shown in Table 6-1. The final decision of the design is to pick up a value which is favorable to all compensator parameters and other specifications. From Table 6-1 assume:

ω_n	C_o	ρ	K_2ka
10	10^4	10	9770
20	5000	20	4960
40	2480	38.4	2460
60	1600	56	1580
80	1200	77	1180
100	922	92	900
200	312	125	365

Table 6-1 Values of ρ, K_2ka and C_o of Example 5-1 for $\zeta = 0.5$

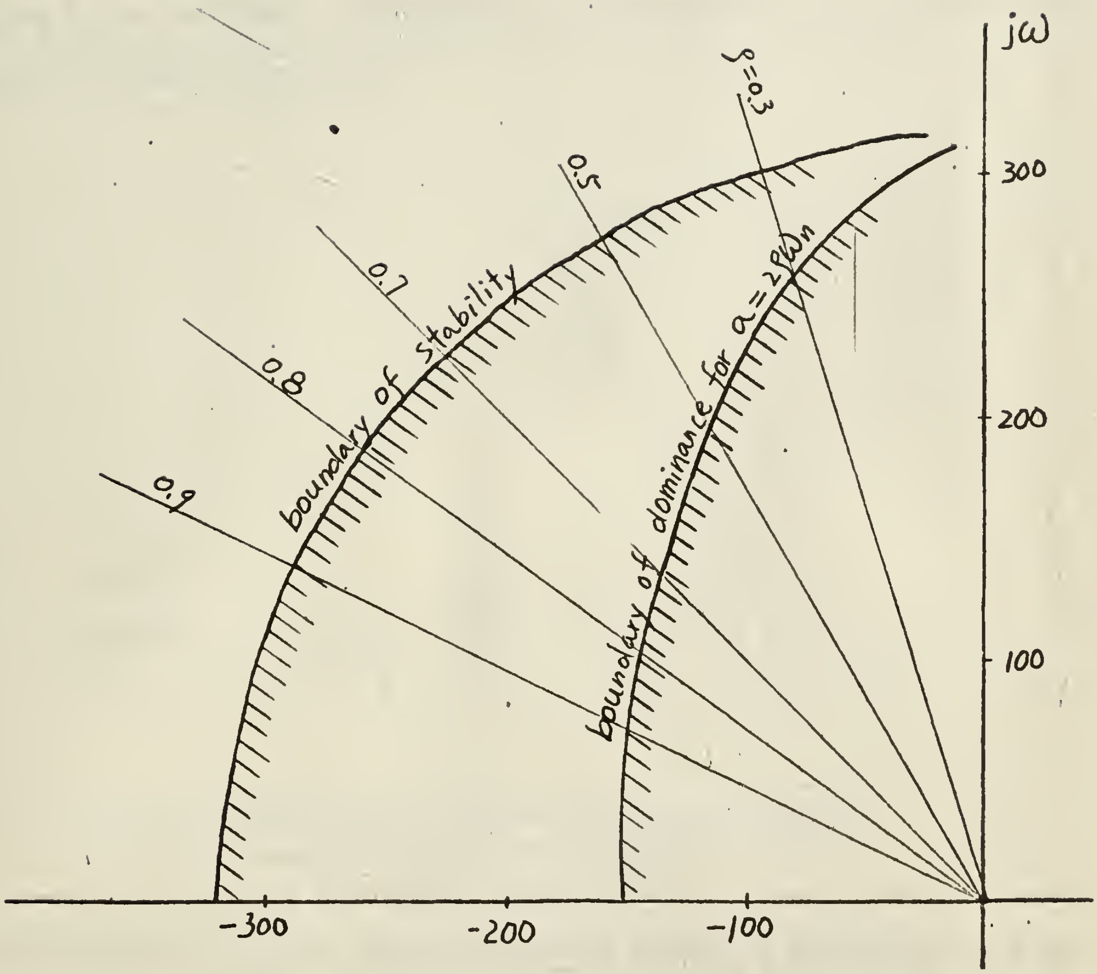


Fig. 6-2 Stable root-region and Dominant root-region of Example 6-1

The dominant roots are chosen to be $\zeta = 0.5$, $\omega_n = 100$, then the third real root is -922, $p = 92$ and $K_2ka = 900$. The closed loop transfer function is

$$\frac{\Theta_c}{\Theta_R} = \frac{K_1 K_2 (s+p)}{(s+s_1)(s+s_2)(s+s_3)}$$

The closed loop pole-zero configuration of the above choice of the dominant roots is shown in Fig. 6-3.

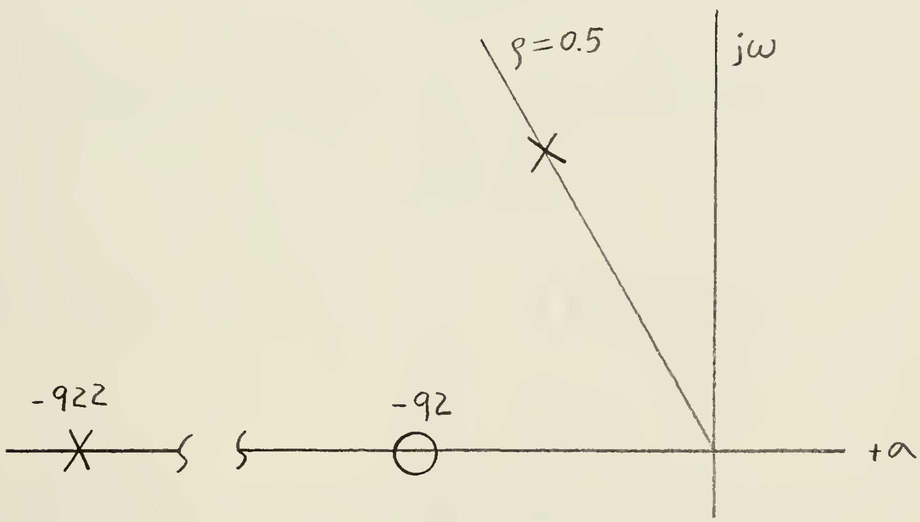


Fig. 6-3 Closed Loop Pole-zero Configuration of Example 6-1 for the dominant roots $\zeta = 0.5$, $\omega_n = 100$

From Fig. 6-3, it can be seen that the system is essentially second order because the real root is far away from the origin. The effect of p to the transient response is also easily visualized from the pole zero configuration.

6-3 Third Order Compensator.

Fig. 6-4 is a block diagram of a third order system with feedback compensator. Assume the forward gain K_1K_2 is fixed from the steady state accuracy and the velocity constant must be greater than a given value K_v.

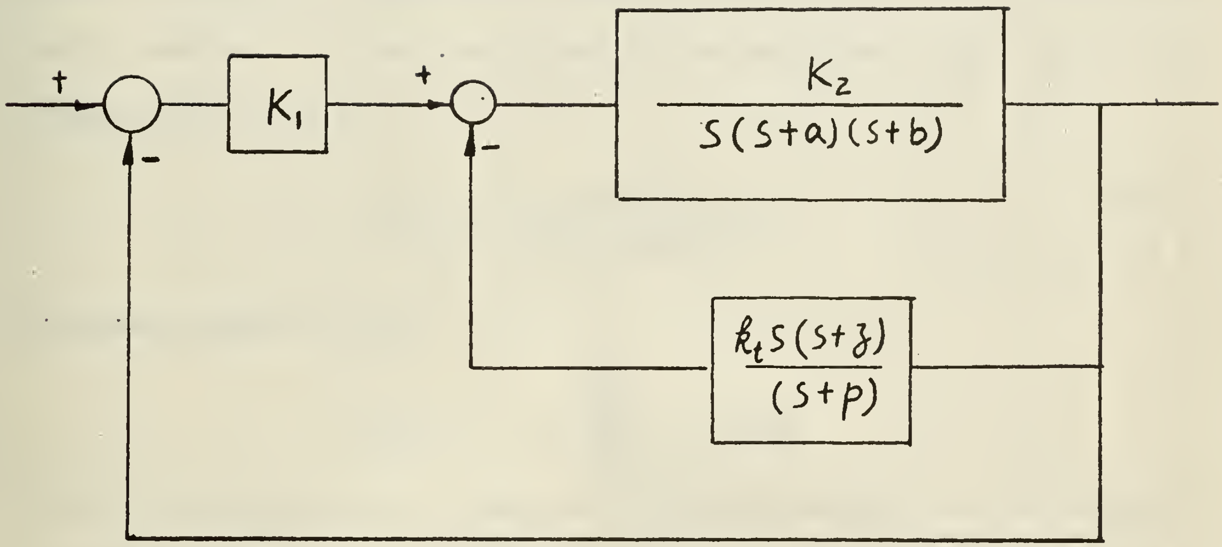


Fig. 6-4 Third Order System with Feedback Compensator.

The characteristic equation of the compensated system is

$$s^4 + (B_2 + p)s^3 + (B_1 + B_2 p + K_2 k_t)s^2 + (B_1 p + K_2 k_t z + B_0)s + B_0 p = 0 \quad (6\text{-}5)$$

where $B_2 = a + b$, $B_1 = ab$, $B_0 = K_1 K_2$

In the above equation, there are three variables, namely k_t, p and z. However, since K_v is specified, those three variables are not independent one to another, they are constrained by the specification K_v. K_v of the compensated system is

$$K_v = \frac{B_0 p}{pB_1 + K_2 k_t z} \quad (6\text{-}6)$$

Rearrange equation (6-6)

$$K_2 k_t z = \frac{pB_0}{K_v} - pB_1$$

Substitute this expression into the coefficient of the first derivative term of equation (6-5), it becomes

$$B_1 p + K_2 k_t z + B_0 = B_1 p + \frac{pB_0}{K_v} - pB_1 + B_0 = \frac{pB_0}{K_v} + B_0$$

Then equation (6-5) becomes

$$s^4 + (B_2 + p)s^3 + (B_1 + B_2 p + K_2 k_t)s^2 + \left(\frac{B_0}{K_v} p + B_0\right)s + B_0 p = 0 \quad (6\text{-}7)$$

The coefficients of characteristic equation now have only two variables, namely k_t and p. The specification of K_v eliminates one of the variables of the compensator. This is always true. The more specifications given, the less the freedom of the compensator. Since there are two variables in the coefficients, two roots are arbitrary. The reduced characteristic equation is of order two and has the form:

$$B_2 + p = 2\zeta\omega_n + C_1 \tag{6-8-1}$$

$$B_1 + B_2 p + K_2 k_t = \omega_n^2 + C_o + 2\zeta\omega_n C_1 \tag{6-8-2}$$

$$\frac{B_o}{K_v} p + B_o = \omega_n^2 C_1 + 2\zeta\omega_n C_o \tag{6-8-3}$$

$$B_o p = \omega_n^2 C_o \tag{6-8-4}$$

where ζ and ω_n are the arbitrary roots while C_o and C_1 are the co-efficient of the reduced characteristic equation. As the four dependent variables C_1, C_o, p and k_t in the above equations are linear functions, it is convenient to express by matrix form as follows:

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ -2\zeta\omega_n & -1 & B_2 & 1 \\ -\omega_n^2 & -2\zeta\omega_n & B_o/K_v & 0 \\ 0 & -\omega_n^2 & 1 & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ C_o \\ p \\ K_2 k_t \end{bmatrix} \quad \begin{bmatrix} 2\zeta\omega_n - B_2 \\ \omega_n^2 - B_1 \\ B_o \\ 0 \end{bmatrix} \tag{6-9}$$

By Cramer's rule, one obtains

$$C_o = \frac{B_o^2 - B_2 B_o \omega_n^2 + 2\zeta\omega_n^3}{2\zeta B_o \omega_n - \frac{B_o}{K_v}\omega_n^2 + \omega_n^4} \tag{6-10}$$

The stability and dominant region are determined by C_o and C_1. Choose dominant roots the parameters $K_2 k_t$, p and z of the compensator are calculated from equations (6-8) and (6-6)

Example 6-2. For the system of Fig. 6-4 given a = 10, b = 30, $K_1 K_2$ = 21,000, $K_v = 60$, damping ratio ζ of the dominant roots be approximately

0.7 and $\omega_n \doteq 10$.

For the given values, $B_2 = 40$

$B_1 = 300$

$B_o = 21{,}000$

Substitute into equation (6-10) for $\zeta = 0.7$

$$C_o = \frac{4.4 \times 10^8 - 8.4 \times 10^5 \omega_n^2 + 1.4\,\omega_n^3}{2.94 \times 10^4 \omega_n - 3.5 \times 10^2 \omega_n^2 + \omega_n^4} \tag{6-11}$$

For $\omega_n = 10$, $C_o = 1321$

From (6-8-4) $p = 6.26$

From (6-8-1) $C_1 = B_2 + p - 2\zeta\omega_n = 32.3$

C_1 and C_2 determine the secondary roots, they must be checked before going further. The reduced characteristic equation is

$$s^2 + 32.2s + 1320 = 0$$

The roots of this equation are $-16.1 \pm j\,32.5$. Compare the real part of these roots to that of the dominant roots ($\zeta\omega_n = 7$), it indicates the presumed dominant roots are dominant.

From (6-8-2)

$$K_2 k_t = \omega_n^2 + C_o + 2\zeta\omega_n C_1 - B_1 - B_2 p = 1323$$

and finally from (6-6)

$$\frac{z}{p} = 2.26\,, \qquad z = 14.15$$

The compensator becomes

$$\frac{1323 \times (s + 14.15)}{K_2\ (s + 6.26)}$$

The calculations above are for one point only. The choice of the dominant roots does not insure a satisfactory result. For more freedom and adjustment, the stability and dominant root region are calculated first as shown in Fig. 6-5. From Fig. 6-5, adjustments of the dominant

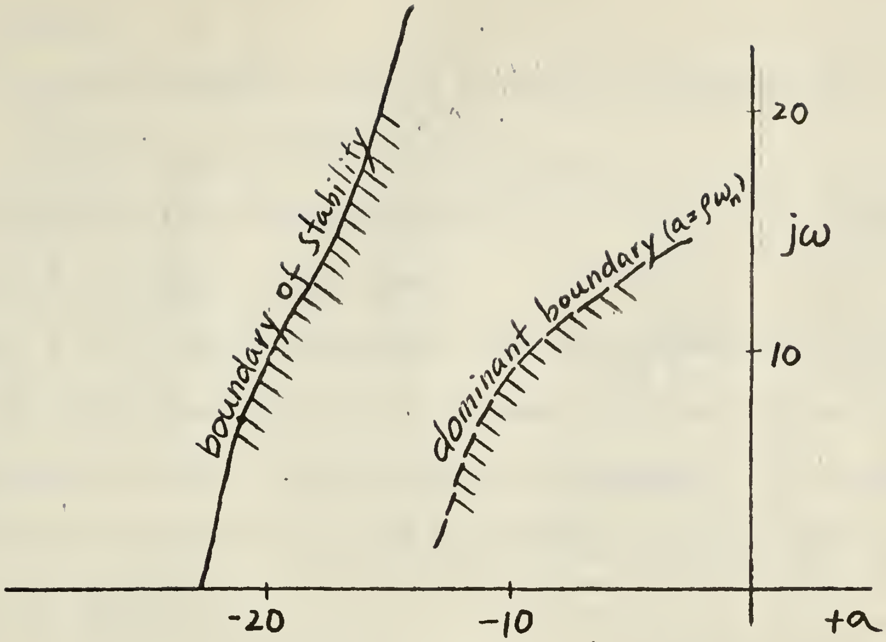


Fig. 6-5. Stability and Dominant Root-regions of Example 6-2

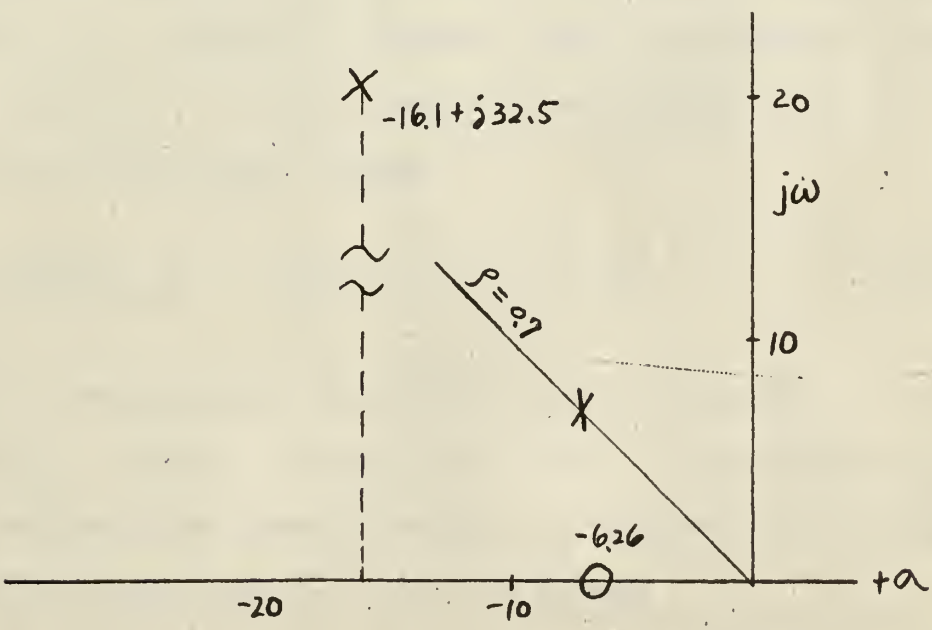


Fig. 6-6. Closed loop pole-zero Configuration of Example 6-2 for $\rho = 0.7$, $\omega_n = 10$.

roots can be made.

The closed loop transfer function is readily formulated as

$$\frac{\theta_c}{\theta_R} = \frac{K_1K_2(S+P)}{(S+S_1)(S+S_2)(S+S_3)(S+S_4)}$$

The closed loop pole-zero configuration for $\zeta = 0.7$, $\omega_n = 10$ (dominant roots) is shown in Fig. (6-6).

Example: 6-3. Given the same plant and the compensator as example 6-2 except the specifications of K_v is relaxed. Here are three free adjustable parameters, three arbitrary roots are available. The controlled characteristic equation is the same as (6-5), the root-coefficient relation are as follows:

$$B_2 + p = 2\zeta\omega_n + S_3 + S_4 \tag{6-12-1}$$

$$B_1 + B_2 p + K_2 k_t = \omega_n^2 + 2\zeta\omega_n(S_3+S_4) + S_3S_4 \tag{6-12-2}$$

$$B_1 p + K_2 k_t z + K = \omega_n^2(S_3+S_4) + 2\zeta\omega_n S_3 S_4 \tag{6-12-3}$$

$$Kp = \omega_n^2 S_3 S_4 \tag{6-12-4}$$

The four dependent variables are K_2K_t, p, z and S_4, the three independent variables are ζ, ω_n and S_3. Assume S_3 and S_4 are complex, and are denoted by ζ' and ω_n'. If ω_n' is chosen as the dependent variable, then from (6-12-1) and (6-12-4) one obtains

$$\omega_n'^2 - \left(\frac{2\zeta' K}{\omega_n^2}\right)\omega_n' - \frac{K}{\omega_n^2}(2\zeta\omega_n - B_2) = 0 \tag{6-13}$$

This equation alone determines all the roots since it contains ζ, ω_n, ζ', and ω_n'. Assume ζ (dominant root) = 0.5, ζ' (secondary root) = 0.7, and substitute the numerical values of K = 21,000, B_1 = 300 and B_2 = 40 into (6-13) and solve for ω_n', one obtains

$$\omega_n' = \frac{1.47\times10^4}{\omega_n^2}\left[1 \pm \sqrt{1 + 9.74\times10^{-5}\omega_n^2(\omega_n - 40)}\right] \tag{6-14}$$

Assume only $10 \leq \omega_n \leq 20$ are of interest, the results of computations from Equations (6-12) and (6-14) are tabulated in Table 6-2.

Notice in Table 6-3 when $\omega_n = 10$ and $\omega_n' = 23.5$, Z is negative.

ω_n	ω_n'	P	$K_2 k_t$	Z	P/Z
10	270	348	62660	10.25	34
	23.5	2.63	575	-7.25	-3.62
15	109.2	128.5	9085	17.2	7.48
	21.4				
20	54.4	56.4	5324	22.3	2.53
	19.6				

Table 6-2 Example 6-3 for $\zeta = 0.5$, $\zeta' = 0.7$

For simple realization of the compensator, the roots of $\omega_n = 15$, $\omega_n' = 109.2$ are chosen. For this choice of the roots, the compensator is

$$G_c = \frac{k_t S(S+17.2)}{(S+128.5)}$$

where $k_t = \frac{K_2 k_t}{K_2}$. If $K_1 = 1$, $k_t = 0.433$. The four roots of the system are $\zeta = 0.5$, $\omega_n = 15$, $\zeta' = 0.7$, $\omega_n' = 109.2$. There is a zero at -128.5. From the closed loop pole-zero configuration, the roots $\zeta = 0.5$, $\omega_n = 15$ have high degree of dominance.

CHAPTER VII

CASCADE COMPENSATION DESIGN

7-1 Single Section:

As cascade compensation is commonly used, the design procedure discussed in Chapter IV, is applied. In Chapter I a third order system was analyzed, here a nth order system in the general case is considered.

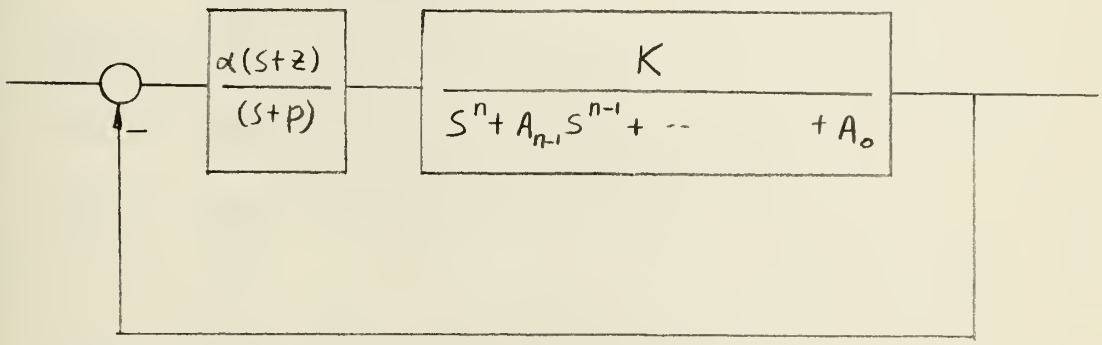


Fig. 7-1 Single Section Cascade Compensation for a nth Order System

Consider the system of Fig. 7-1. Assume the characteristic equation of the uncompensated system is

$$S^{n} + B_{n-1} S^{n-1} + B_{n-2} S^{n-2} + \cdots + B_2 S^2 + B_1 S + B_0 = 0 \qquad (7\text{-}1)$$

For specified forward gain K and error constant (K_v, K_a, etc.) the three compensator parameters Z, p and α are not independent one to another, but related, namely

$$\alpha = \frac{p}{z} \qquad (7\text{-}2)$$

The characteristic equation of the compensated system is

$$S^{n+1} + (B_n + p) S^{n} + \cdots + (B_i + B_{i+1} p) S^{i} + \cdots + (B_0 \alpha + B_1 p) S + B_0 p = 0 \qquad (7\text{-}3)$$

In equation (7-3), there are two variables p and α in the coefficients, then two roots are arbitrary. The reduced characteristic equation is of order (n-1), and has the following form.

$$S^{n-1} + C_{n-2} S^{n-2} + \cdots + C_1 S + C_0 = 0$$

If the two arbitrary roots are presumed to be the dominant roots of the compensated system and are expressed by ζ and ω_n then the root-coefficient relations of equation (7-3) are:

$$\begin{aligned}
B_0 p &= C_0 \omega_n^2 \\
B_0 \alpha + B_1 p &= C_1 \omega_n^2 + C_0 2\zeta\omega_n \\
B_1 + B_2 p &= C_2 \omega_n^2 + C_1 2\zeta\omega_n + C_0 \\
B_2 + B_3 p &= C_3 \omega_n^2 + C_2 2\zeta\omega_n + C_1 \\
&\vdots \\
B_{i-1} + B_i p &= C_i \omega_n^2 + C_{i-1} 2\zeta\omega_n + C_{i-2} \\
&\vdots
\end{aligned} \tag{7-4}$$

The N+1 dependent variables (p , α , C_o, $C_1 \ldots C_{n-2}$) are linear functions, then equation (7-4) can be expressed in matrix form

$$\begin{bmatrix}
B_n & 0 & 0 & 0 & \cdots & 0 & -1 \\
B_{n-1} & 0 & 0 & \vdots & & -1 & -2\zeta\omega_n \\
B_{n-2} & 0 & \vdots & \vdots & & -2\zeta\omega_n & -\omega_n^2 \\
B_{n-3} & 0 & \vdots & \vdots & & -\omega_n^2 & \vdots \\
\vdots & \vdots & \vdots & \vdots & & 0 & \vdots \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
\vdots & \vdots & 0 & 0 & & \vdots & \vdots \\
\vdots & 0 & 0 & -1 & & \vdots & \vdots \\
B_2 & 0 & -1 & -2\zeta\omega_n & & \vdots & \vdots \\
B_1 & B_0 & -2\zeta\omega_n & -\omega_n^2 & & \vdots & \vdots \\
B_0 & 0 & -\omega_n^2 & 0 & \cdots & 0 & \vdots
\end{bmatrix}
\begin{bmatrix} p \\ \alpha \\ C_0 \\ C_1 \\ C_2 \\ \vdots \\ C_{n-2} \end{bmatrix}
=
\begin{bmatrix} 2\zeta\omega_n \\ \omega_n^2 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}
-
\begin{bmatrix} B_{n-1} \\ B_{n-2} \\ \vdots \\ B_2 \\ B_1 \\ 0 \\ 0 \end{bmatrix} \tag{7-5}$$

By Cramer's rule one obtains

$$\frac{-B_0\left[\varphi_1 B_1 + \varphi_2 B_2 \omega_n + \varphi_3 B_3 \omega_n^2 + \cdots \qquad + \varphi_i B_i \omega_n^{i-1}\right.}{B_0 + \varphi_1 B_2 \omega_n^2 + \varphi_2 B_3 \omega_n^3 + \varphi_3 B_4 \omega_n^4 + \cdots + \varphi_i B_{i+1} \omega_n^{i+1} + \cdots} \tag{7-6}$$

where

$$\begin{aligned} \varphi_1 &= -1 \\ \varphi_2 &= 2\zeta \\ \varphi_3 &= 1-4\zeta^2 \\ \varphi_4 &= -4\zeta+8\zeta^3 \\ &\vdots \\ \varphi_i &= -\left[2\zeta\varphi_{i-1}(\zeta) - \varphi_{i-2}(\zeta)\right] \\ &\vdots \end{aligned} \tag{7-7}$$

The other dependent variables can be expressed in the same way but this is not necessary as far as computational purposes are concerned. When C_o has been calculated, the other variables are readily evaluated from equations (7-4). Choose ζ , evaluate the stability and dominant root interval of ω_n . Within the dominant root region, choose ω_n and evaluate ρ and α . If the network is realizable, the design wrok is completed.

Example 7-1. Given $G = \frac{400}{s^2}$ and K_a is not to be reduced. Then the characteristic equation of the uncompensated system is

$$s^2 + 400 = 0$$

$B_2 = 1$, $B_1 = 0$, $B_o = 400$, $B_i = 0$ where $i > 2$

The order of the compensated system is three and that of the reduced characteristic equation is one. Then

$$C_1 = 1, \quad C_i = 0 \quad \text{when } i > 1$$

From Equation 7-4:

$$
\begin{aligned}
B_0 p &= C_0 \omega_n^2 \\
B_0 \alpha + B_1 p &= C_1 \omega_n^2 + C_0 2\zeta\omega_n \\
B_1 + B_2 p &= C_1 2\zeta\omega_n + C_0
\end{aligned}
\tag{7-8}
$$

Substitute the values of B's into the above equations, obtains

$$
\begin{aligned}
400p &= C_0 \omega_n^2 \\
400\alpha &= \omega_n^2 + 2\zeta\omega_n C_0 \\
p &= 2\zeta\omega_n + C_0
\end{aligned}
\tag{7-9}
$$

From equation (7-6) and (7-7)

$$C_0 = \frac{B_0(B_1 - 2\zeta\omega_n)}{B_0 - \omega_n^2} \tag{7-10}$$

Substitute the values of B's, obtain

$$C_0 = \frac{800\zeta\omega_n}{\omega_n^2 - 400} \tag{7-11}$$

For stability the criterion is $C_0 > 0$, namely

$$\frac{800\zeta\omega_n}{\omega_n^2 - 400} > 0$$

The numerator never becomes negative, then the inequality becomes

$$\omega_n^2 > 400 \tag{7-12}$$

Define the dominant roots such that the real part of the secondary roots must be at least twice that of the dominant roots, then

$$C_0 > 2\zeta\omega_n \,, \qquad \frac{800\zeta\omega_n}{\omega_n^2 - 400} > 2\zeta\omega_n$$

Then the criterion for dominant roots is

$$400 < \omega_n^2 < 800$$

Those regions are shown in Fig. 7-2.

Within the dominant root region, for $\zeta = 0.5$, and $\zeta = 0.6$, the compensator parameters are computed from equation (7-8) for different values of ω_n and tabulated in Table 7-1. Table 7-1 includes not only the compensator parameters, but also all the roots of the system for different choice of the compensator. Assume the dominant roots are chosen as $\zeta = 0.6$, $\omega_n = 30$, then from Table 7-1, $C_o = 28.8$, $p = 64.8$, $\alpha = 4.41$, $z = \frac{p}{\alpha} = 14.8$, The compensator is $G_c = \frac{4.41(s+14.8)}{(s+64.8)}$

	$\zeta = 0.5$			$\zeta = 0.6$		
ω_n	C_o	p	α	C_o	p	α
22	220	242	13.2	264	291	14
24	54.6	78.5	4.72	67.7	94.3	5.34
26	37.7	63.4	4.12	45.3	76.1	4.6
28	29.2	57.2	4	35.1	68.5	4.41
30	24	54	4.05	28.8	64.8	4.41
32	20.5	54.6	4.2	24.6	65.6	4.53
34	19.3	55.7	4.53	23.2	66.8	4.87

Table 7-1 Computation Result of Example 7-1

The closed loop-pole-zero configuration for this choice of the dominant roots is shown in Fig. 7-3(a). In Fig. 7-3(a) it can be visualized that the peaking time is small but the overshoot may be too large because the zero is too close to the origin[6]. If $\zeta = 0.6$, $\omega_n = 24$, are chosen as the dominant roots, then from Table 7-1, $C_o = 67.7$, $p = 94.3$, $\alpha = 5.34$ and $z = 17.7$. The compensator is

$$G_c = \frac{5.34(s+17.7)}{(s+94.3)}$$

The pole-zero configuration for this choice of the dominant roots is shown in Fig. 7-3(b). For this choice of the dominant roots the peaking time is still small, but the overshoot is reduced.

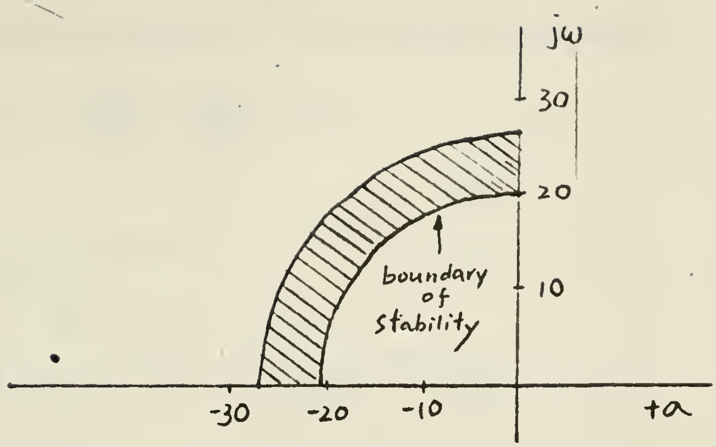


Fig. 7-2
Stable and Dominant Region of Example 7-I

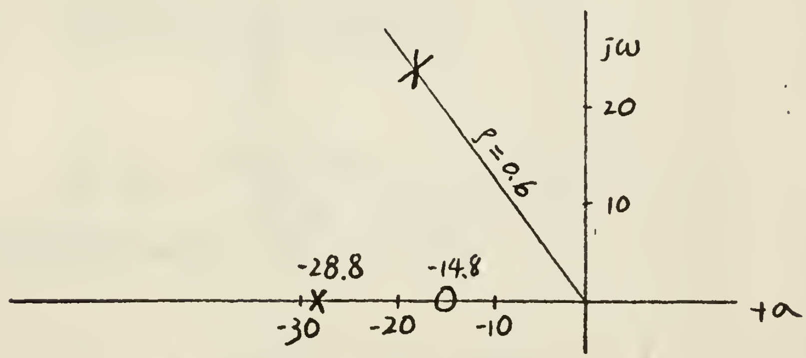


Fig. 7-3(a)

Closed Loop Pole-zero Configuration for Dominant Roots $\rho = 0.6$, $\omega_n = 30$ for Example 7-1.

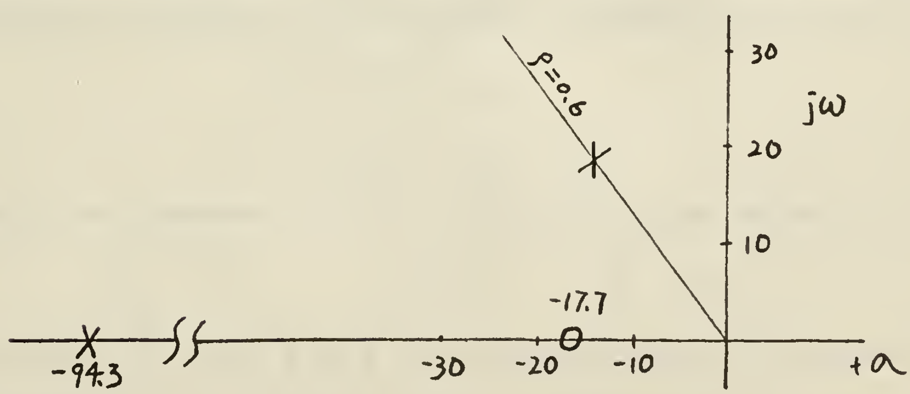


Fig. 7-3(b)
Closed Loop Pole-zero Configuration for Dominant Roots $\rho = 0.6$, $\omega_n = 24$ of Example 7-1.

Example 7-2: Given $G = \frac{300}{s(s+2)(s+5)}$ and K_v is not to be reduced. The characteristic equation of the uncompensated system is

$$s^3 + 7s^2 + 10s + 300 = 0$$

From Equation 7-4:

$$\begin{aligned} 7 + p &= C_1 + 2\zeta\omega_n \\ 10 + 7p &= \omega_n^2 + C_0 + 2\zeta\omega_n C_1 \\ 10p + 300\alpha &= C_1\omega_n^2 + 2\zeta\omega_n C_0, \quad 300p = C_0\omega_n^2 \end{aligned} \tag{7-12}$$

From Equation (7-6) and (7-7)

$$C_0 = \frac{-300\left[-10 + 14\zeta\omega_n + (1-4\zeta^2)\omega_n^2\right]}{300 - 7\omega_n^2 + 2\zeta\omega_n^3} \tag{7-13}$$

For ζ = 0.5

$$C_0 = \frac{(7\omega_n - 10)\,300}{7\omega_n^2 - \omega_n^3 - 300}$$

For = 0.3

$$C_0 = \frac{300(0.64\omega_n^2 + 4.2\omega_n - 10)}{7\omega_n^2 - 0.6\omega_n^3 - 300}$$

To determine the stability region, the criterion is

$$C_0 > 0 \qquad \text{and} \qquad C_1 > 0 \tag{7-14}$$

Define the dominant roots by a = $\zeta\omega_n$, from Table 2-7 of Chapter II, the transformed coefficients of the reduced characteristic equation are:

$$\begin{aligned} D_1 &= C_1 - 2a \\ D_0 &= C_0 - C_1 a + a^2 \end{aligned}$$

The criterion for dominant roots by this definition of dominant roots is:

$$D_1 > 0 \qquad \text{and} \qquad D_0 > 0 \tag{7-15}$$

Computations for ζ = 0.3 and ζ = 0.5 from Equations (7-13) and (7-12) are shown in Table 7-2 and Table 7-3 respectively.

ω_n	C_0	p	C_1	α	D_1	D_0
0.2	9.13	0.001217	7	0.0046	+	+
0.4	8.22	0.0044	6.88	0.0101	+	+
0.6	7.25	0.0084	6.81	0.0166	+	+
0.8	6.34	0.0135	6.49	0.0235	+	+
1.0	5.28	0.0176	6.41	0.0313	+	+
1.2	4.16	0.02	6.27	0.0397	+	+
1.4	2.97	0.0193	6.21	0.049	+	+
1.6	1.71	0.0146	6.07	0.057	+	+
1.8	0.397	0.0043	8.86	0.097	+	+
1.9	-0.312	-0.0038	4.73		+	-

Table 7-2 $\zeta = 0.3$ for Example 7-2

ω_n	C_0	p	C_1	α	D_1	D_0
0.6	5.84	0.007	6.4	0.019	+	+
0.8	4.46	0.0095	6.2	0.0191	+	+
1.0	3.07	0.01022	6.01	0.0216	+	+
1.1	2.24	0.00904	6.02	0.0300	+	-
1.2	1.56	0.00748	6.05	0.0322	+	-
1.3	0.87	0.0049	5.75	0.036	+	-
1.4	0.1925	0.00125	5.61	0.0375	+	-
1.5	-0.48	-0.0036	5.46	0.0383	+	-

Table 7-3 $\zeta = 0.5$ for Example 7-2

From the criteria (7-14) and (7-15), the stability and dominant region for $\zeta = 0.3$ and $\zeta = 0.5$ are determined from the computation tables.

	Stability	Dominant ($a = \zeta\omega_n$)
$\zeta = 0.3$	$\omega_n < 1.8$	$\omega_n < 1.8$
$\zeta = 0.5$	$\omega_n < 1.4$	$\omega_n < 1.05$

Since this computation is a progressing process, when the stability and dominant root region have been located, the compensator parameters also have been calculated already. The computation tables not only include the compensator parameters, but also the secondary roots. The coefficients (D's) of the transformed equation are shown signs only in the compensation tables, because in this case only signs are needed to determing the dominant region.

Assume the dominant roots are chosen as $\zeta = 0.3$, $\omega_n = 1.0$, then from Table 7-2, $C_o = 5.28$, $C_1 = 6.41$, $p = 0.0176$, $\alpha = 0.0313$ and $z = \frac{p}{\alpha} = 0.563$. The reduced characteristic equation is

$$S^2 + 6.41S + 5.28 = 0$$

Then the secondary roots are S = -5.44, -0.97. The closed loop pole-zero configuration is shown in Fig. 7-4(a). In Fig. 7-4(a), all the secondary roots are to the left of the dominant roots.

Consider Table 7-2 and 7-3, they indicate $\alpha < 1$, this implies the compensator is a lag network. For a lag network the zero is close to the origin. Then a root to the right of the dominant roots is allowed. This dipole near the origin has negligible effect on the transient response if the root is close enough to the zero. In this sense, the definition of the dominant roots may be adjusted as such that all the secondary roots must be to the left of the dominant roots except the root which is close to the zero.

In this example: if $\zeta = 0.5$, $\omega_n = 1.2$ is chosen as the dominant roots, from Table 7-3, $C_o = 1.56$, $C_1 = 7.25$ $p = 0.00748$, $\alpha = 0.035$ and $z = \frac{p}{\alpha} = 0.214$. The reduced characteristic equation is

$$S^2 + 7.25S + 1.56 = 0$$

Then the secondary roots are -7.03 and -0.225. The closed loop pole-zero configuration is shown in Fig. 7-4(b). Compare the pole-zero configurations of Fig. 7-4(a) and Fig. 7-4(b), it can be easily seen that the scheme of Fig. 7-4(b) has better performance than that of scheme Fig. 7-4(a).

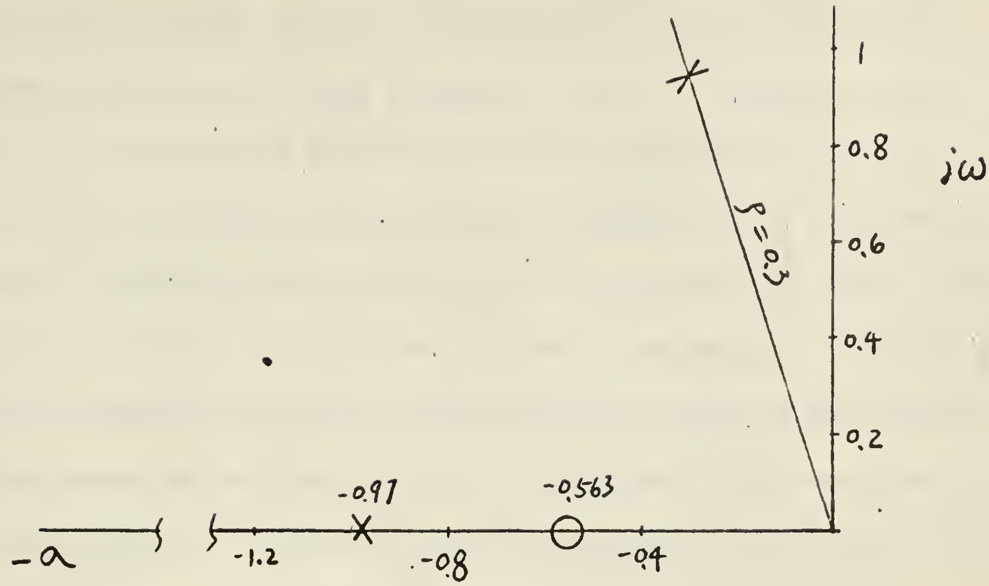


Fig. 7-4(a) Closed Loop Pole-zero Configuration for the Dominant Roots $\rho = 0.3$, $\omega_n = 1$ of Example 7-2

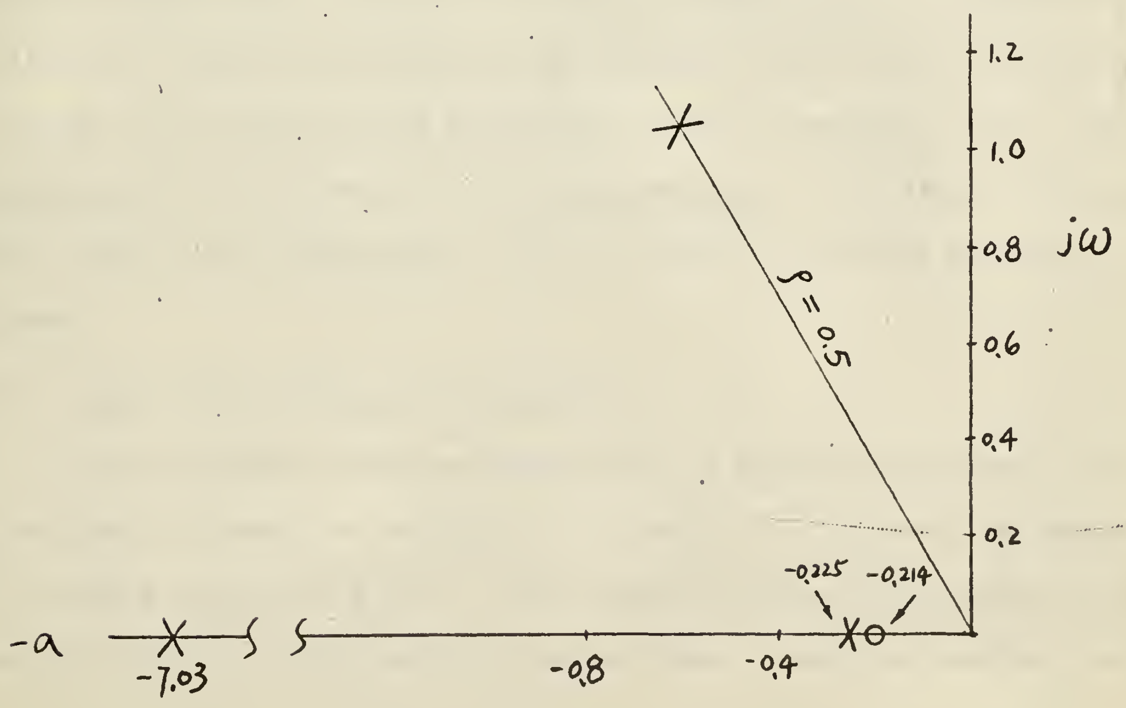


Fig. 7-4(b) Closed Loop Pole-zero Configuration for a Dipole Near the Origin of Example 7-2. Dominant Roots are $\rho = 0.5$, $\omega_n = 1.2$.

7-2 Stability Boundary Lines.

It has been pointed out that the stability boundary lines for a cascade compensator are the same boundary lines of relocation zones in Reference 7. In the single section zone if the compensator is to be realized by an R-C circuit with pole-zero on negative real axis, then the necessary and sufficient condition in all the roots are within that zone. That is in the single section zone the compensator, $G_c = \frac{\alpha(s+z)}{(s+p)}$ always can be realized by an R-C circuit for any choice of the roots providing those roots are within that zone. In the multiple zone, the roots being in that zone is a necessary condition but not a sufficient condition.

For higher order systems there may be several zones for a single section compensator. But for dominance, only the zone which is the closest to the $j\omega$ axis is of interest. It has been proved that putting a pair of complex roots within the lag area and close to the stability boundary lines (uncompensated root-loci) implies a closed loop dipole close to the origin. By the same reason, putting a pair of complex roots within the lead area and close to the boundary line (pseudo-root loci) implies all secondary roots are far away from the dominant roots. The question of how far away and how close they are, the dominant root boundary lines of the definition a = $\zeta\omega_n$ serves a guide for both conditions.

7-3 Double Section Cascade Compensator.

If the dominant roots are expected to be within the multiple section zone, at least two sections are necessary. In general, the design is carried out by two steps. First choose an intermediate dominant root which is close to the stability boundary lines. Form the transfer function

obtained by the intermediate dominant roots, then repeat the process. The working formulas for the open loop transfer function with one zero in it have been derived in Appendix 2 for an original nth order system. The design by intermediate dominant roots has two advantages. First the quadratic term as discussed in Chapter 3, equations (3-3) is avoided. Secondly the compensator can be designed as a double lead section, a double lag section or a lead-lag network. The intermediate dominant roots should be chosen close to the final dominant roots because the regions which are close to the dividing lines (both uncompensated root-loci and the pseudo root-loci) are dominant root regions.

Example 7-3: Given $G_T = \frac{400}{s^2}$, the system is expected to operate on a pair of dominant roots $\zeta = 0.7$, $\omega_n = 20$, and K_a is not to be reduced.

From Example 7-1 and Fig. 7-2, $\omega_n = 20$ is on the stability boundary line, then double section is needed. From Table 7-1, choose the intermediate dominant roots $\zeta = 0.5$, $\omega_n = 22$, which is close to the final dominant roots $\zeta = 0.7$, $\omega_n = 20$. The transfer function becomes

$$G = \frac{5280(s+18.5)}{s^2(s+242)} \tag{7-16}$$

From Appendix 2 equation A-2-4, the four equations are

$$p - C_1 = 2\zeta\omega_n - B_2 \tag{7-17-1}$$

$$B_2 p + \frac{B_0}{C}\alpha - C_0 - 2\zeta\omega_n C_1 = \omega_n^2 - B_1 + \frac{B_0}{C_1} \tag{7-17-2}$$

$$B_1 p + B_0\alpha - 2\zeta\omega_n C_0 - \omega_n^2 C_1 = 0 \tag{7-17-3}$$

$$B_0 p - \omega_n^2 C_0 = 0 \tag{7-17-4}$$

From A-2-6:

$$C_o = \frac{B_o\left[B_1 - 2\rho B_2\omega_n - (1-4\rho^2)\omega_n^2\right] - \frac{B_o}{C}\left[B_o - B_2\omega_n^2 + 2\rho\omega_n^3\right]}{\left[B_o - B_2\omega_n^2 + 2\rho\omega_n^3\right] + \frac{\omega_n}{C}\left[-2\rho B_o + B_1\omega_n - \omega_n^3\right]} \tag{7-18}$$

where B_o, B_1 and B_2 are the coefficients of the characteristic equation of the open loop transfer function (7-16), (-C) is the open loop zero. The stability boundary lines are shown in Fig. 7-5a by the rules of reference 7. From Fig. 7-5, it indicates that the roots of $\rho = 0.7$, $\omega_n = 20$, are within the single lead section zone and close to the pseudo root-loci. Then they are expected to be dominant. Substitute C = 185, B_o = 97500, B_1 = 5280, B_2 = 242, $\rho = 0.7$, $\omega_n = 20$ into equation (7-18), one obtains

	$C_o = 6340$
From (7-17-4)	$p = 26$
From (7-17-1)	$C_1 = 240$
From (7-17-3)	$\alpha = 1.4$
Then	$Z = 18.6$

The secondary roots are determined from the equation

$$S^2 + 240S + 6340 = 0 \tag{7-19}$$

The solution of (7-19) is -210, -30.

The compensator for the second stage is $G_c' = \dfrac{1.4\,(S+18.6)}{(S+26)}$

The overall compensator is

$$G_c = \frac{18.5\,(S+18.6)^2}{(S+242)\,(S+26)}$$

The closed loop pole-zero configuration is shown in Fig. 7-5b. From Fig. 7-5b it can be visualized that the response is speeded up by the two zeros while the overshoot is small because of the approximately equal

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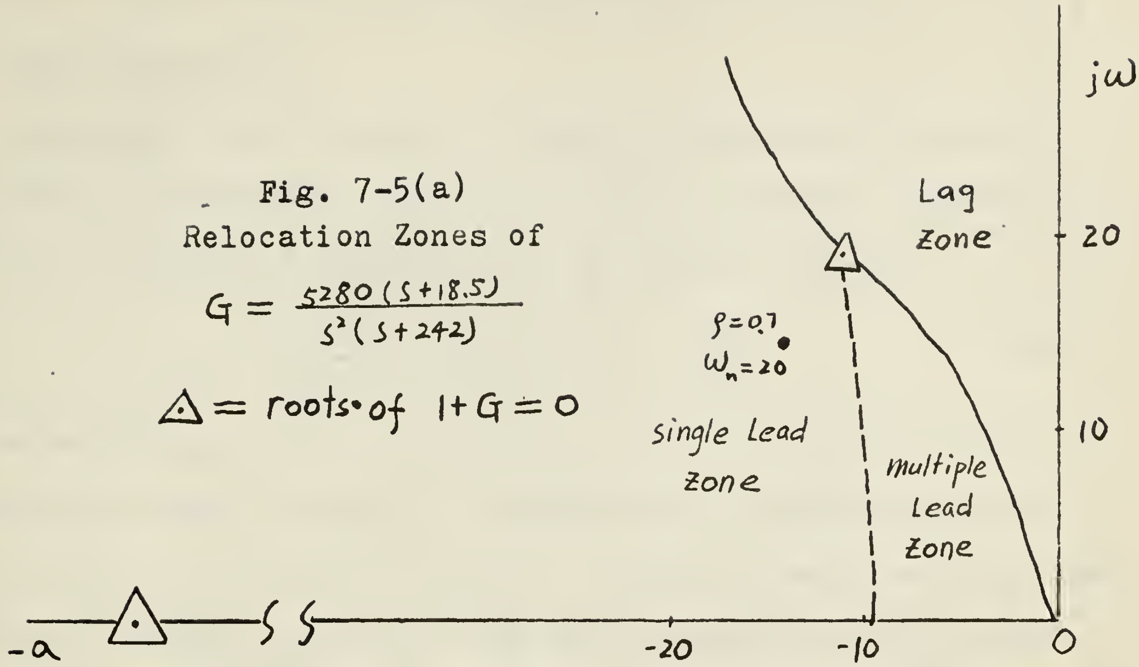

Fig. 7-5(a)
Relocation Zones of

$$G = \frac{5280(s+18.5)}{s^2(s+242)}$$

$\triangle$ = roots of $1+G=0$

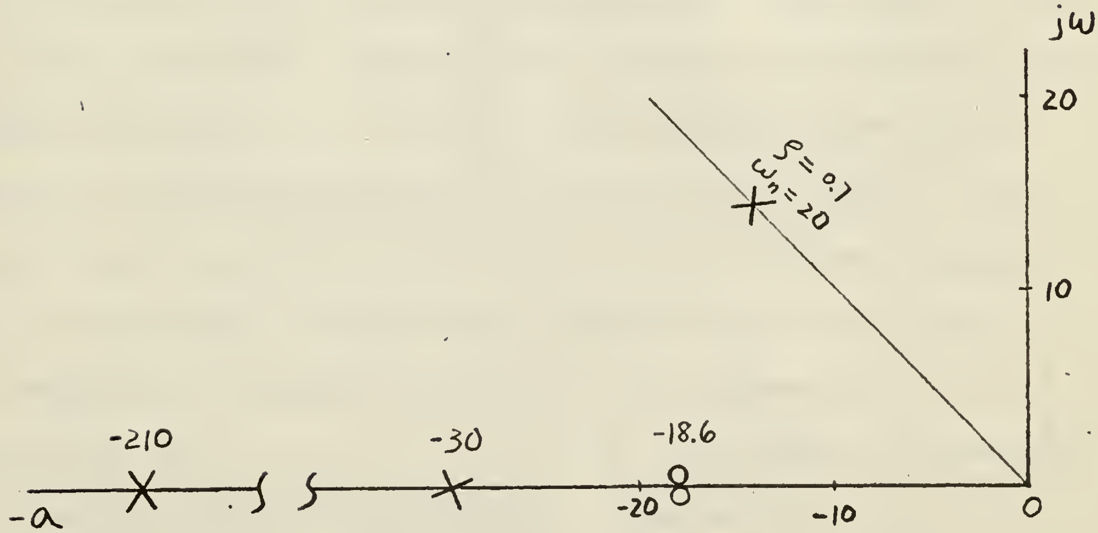

Fig. 7-5(b) Final Closed Loop Pole-zero Configuration of Example 7-3

distances of the zeros and the dominant roots to the origin.

7-4 Mixed Compensators.

For a single section cascade or feedback compensator it has been shown that the adjustable parameters are linearly dependent (Matrices 6-9 and 7-5). In general a compensator has the transfer function

$$G_c = \frac{a_p s^p + a_{p-1} s^{p-1} + \cdots + a_0}{b_q s^q + b_{q-1} s^{q-1} + \cdots + b_0} \tag{7-20}$$

The coefficients, a_0, a_1, -- a_p and b_0, b_1, -- b_q are linearly dependent for either a cascade or a feedback path. The solutions given a set of roots are always real numbers. If there are more than one compensator and in different loops, the adjustable parameters are not linearly dependent. The solutions for a given set of roots may be complex numbers and consequently the compensators are not physically realizable. However, the subset of roots which gives the solution of real numbers can be found if such a subset exists in the stable root region. Note the coefficients of the reduced characteristic equations are always real numbers for any choice of the roots and for any type of compensators. The reason for this fact is due to the complex conjugate roots. Because of this fact the stable root region and dominant root region are always defined if they exist. The physically adjustable parameters also appear as real numbers in some transformations for any choice of the roots, but the individual parameters may not be real numbers. Consider Fig. 7-6 which consists of a single section compensator in cascade and another in the feedback path, G is the given plant and z_1, z_2, p_1, p_2 and k_t are adjustable parameters. From a signal flow graph, the characteristic equation is

$$(s+p_1)(s+p_2)Q + k_t(s+z_1)(s+p_2)P + k_a s(s+z_2)(s+p_1)P = 0 \tag{7-21}$$

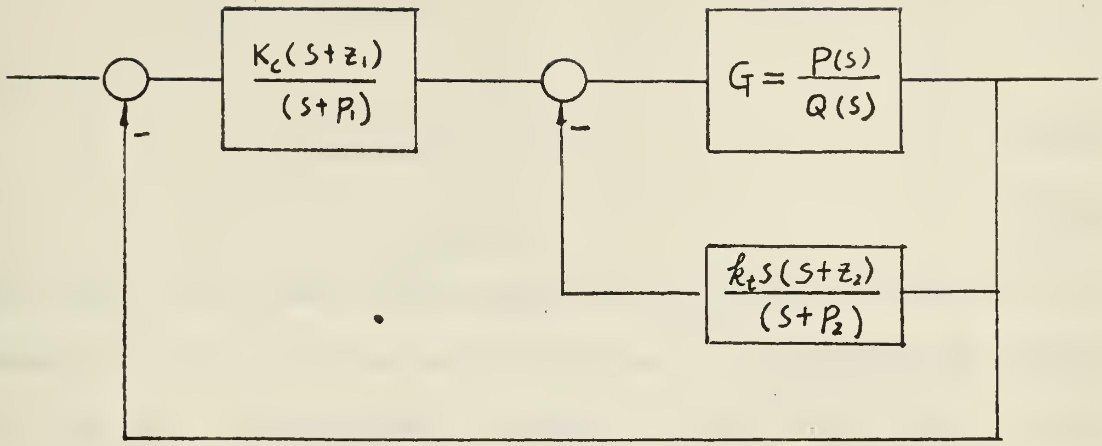

Fig. 7-6 Mixed Compensators

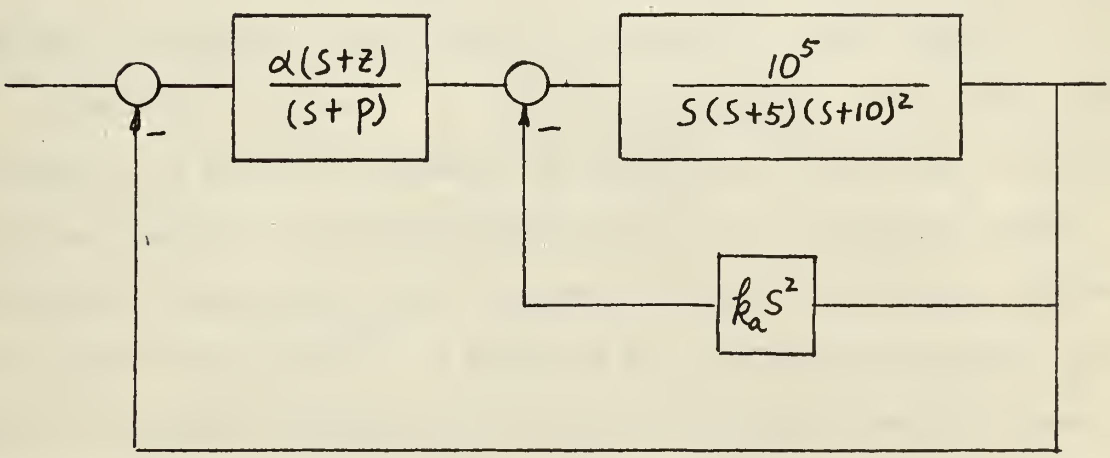

Fig.7-6(a) Mixed Compensation of Example 7-4

Consider the first term of (7-21), it can be expressed as follows:

$$(s+p_1)(s+p_2)Q = [s^2 + (p_1+p_2)s + p_1 p_2]Q \tag{7-22}$$

Let

$$p_1 + p_2 = b \tag{7-23}$$

$$p_1 p_2 = a \tag{7-24}$$

where b and a are always real numbers for any choice of roots of (7-21). However, p_1 and p_2 are not necessarily real. To find the real solution of p_1 and p_2, the roots must be subject to some restrictions. In this case it is

$$b^2 > 4a \tag{7-25}$$

The inequality (7-25) defines the realizable region. For a passive network, additional constraints are necessary, that is

$$b > 0 \qquad a > 0 \tag{7-26}$$

The same constraints of the roots can be applied to other cases.

Example 7-4. Given $G = \dfrac{10^5}{s(s+5)(s+10)^2}$ The velocity error constant K_v is not to be reduced. The band width is less than 20 rad/sec. From the specifications dominant roots of $\zeta \doteq 0.5$, $\omega_n \doteq 10$ are chosen. For cascade compensation alone, more than two lead sections are required by the Ross Warren method[7]. A scheme of Fig. 7-6(a) is attempted. Because of the gain requirement, only second derivative feedback is used and the parameters α, z and p are related by the following relation

$$\alpha = \frac{p}{z} \tag{7-27}$$

There are three free parameters (α, p, k_a), three roots are artitrary. The constant controlled characteristic equation is

$$s^5 + (25+p)s^4 + (200 + 500p + 10^5 k_a)s^3 + (500 + 200p + 10^5 k_a p)s^2$$
$$+ (500p + 10^5\alpha)s + 10^5 p = 0 \tag{7-28}$$

Let $(-S_1)$, $(-S_2)$ and $(-S_3)$ be the arbitrary roots and define the following terms for simplification.

$$F_1 = S_1 + S_2 + S_3 \tag{7-29-1}$$

$$F_2 = S_1 S_2 + S_2 S_3 + S_3 S_1 \tag{7-29-2}$$

$$F_3 = S_1 S_2 S_3 \tag{7-29-3}$$

The root-coefficient relations are:

$$500 + p = F_1 + C_1 \tag{7-30-1}$$

$$200 + 500p + 10^5 k_a = C_o + C_1 F_1 + F_2 \tag{7-30-2}$$

$$25 + 200p + 10^5 k_a p = C_o F_1 + C_1 F_2 + F_3 \tag{7-30-3}$$

$$25p + \alpha 10^5 = C_o F_2 + C_1 F_3 \tag{7-30-4}$$

$$10^5 p = C_o F_3 \tag{7-30-5}$$

Where C_1 and C_o are the coefficients of the reduced characteristic equation which defines the constrained roots $(-S_4)$ and $(-S_5)$ by the equation

$$S^2 + C_1 S + C_o = 0 \tag{7-31}$$

From equations (7-30-1), (7-30-2), (7-30-3) and (7-30-5), C_o is obtained:

$$F_3 (F_1 F_3 - 25 F_3 + 10^5) C_o^2 - 10^5 [F_1^2 F_3 - 25 F_1 F_3 + 10^5 F_1] C_o$$
$$+ 10^{10} [F_1 F_2 - 25 F_2 - F_3 + 500] = 0 \tag{7-32}$$

Assume $(-S_1)$ and $(-S_2)$ are the dominant roots and denoted by ρ and ω_n. $(-S_3)$ is arbitrary and must be greater than $\rho\omega_n$. Assume $S_3 = 2\rho\omega_n$

For $\rho = 0.5$, $\omega_n = 10$ from (7-29)

$F_1 = 20$

$F_2 = 200$

$F_3 = 1000$

Substitute into (7-32) one obtains,

$$C_o = +2080, \quad -31.6$$

Take $C_o = 2080$

From (7-30-5) $\quad \rho = 20.8$

From (7-30-1) $\quad C_1 = 25.8$

From (7-30-2) $\quad 10^5 k_\alpha = 2076$

From (7-30-4) $\quad \alpha = 4.32$

The reduced characteristic equation is

$$s^2 + 25.8s + 2080 = 0$$

The constrained roots are $-S_4, \ -S_5 = -12.9 \pm j\,43.5$

The five roots are:

$$-S_1, -S_2 = -5 \pm j\,8.66$$

$$-S_3 = -10$$

$$-S_4, -S_5 = -12.9 \pm j\,43.5$$

The compensator is

$$G_c = \frac{4.32\,(s + 4.83)}{s + 20.8}$$

$$k_\alpha = 0.02078$$

The closed loop pole-zero configuration is shown in Fig. 7-7. For other choices of the arbitrary roots, better results can be obtained.

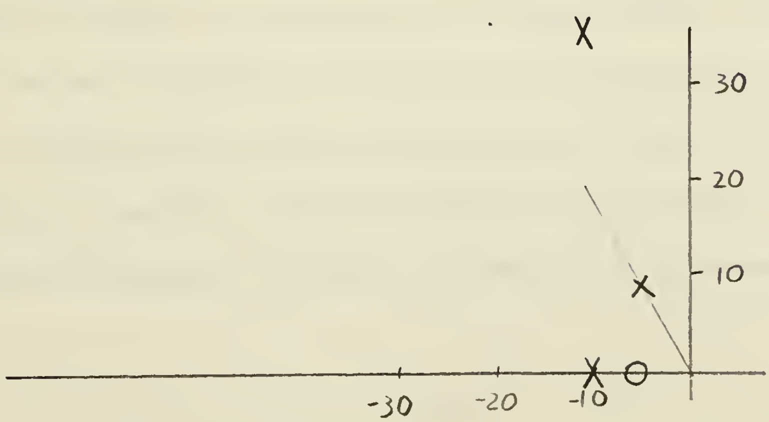

Fig. 7-7 Closed Loop Pole-zero Configuration of Example 7-4.

VIII. CONCLUSIONS

8-1 Summary of Results.

A new method for design of control systems in the time domain has been described. The method is applicable to any system configuration and is able to handle any number of variable parameters. The design procedure is based upon a set of simultaneous algebraic equations, and is well suited to implementation by digital computer. The method is flexible and has more freedom in design than other methods. The values of adjustable parameters are computed from analytical expressions, so any desired degree of accuracy can be obtained.

The main advantage of this method over those methods available in the literature lies in the fact that the method is a direct syntheses from the closed loop pole-zero configuration. No trial and error is needed. The specifications and restrictions of the compensator confine the arbitrary roots within a region on the S-plane, once this region or regions have been determined, the rest of the design work is just a computation. The compensator obtained by this computation guarantee that the specifications and restrictions are satisfied.

The main difference between this method and other available methods is that the domain of the system can be any number of variables. Theoretically, the more variables available, the more design freedom one has. In the root-locus method, only one parameter is varied others must be fixed. In Mitrovic's method, only two parameters are varied. This method is able to handle any number of variable parameters simultaneously.

8-2 General Description of the Method.

The design procedure can be summarized as shown in Fig. 8-1. The analytical expressions of the dependent variables of commonly used compensators have been derived in the previous chapters. In general for the case in which coefficients of the characteristic equations as variables, the dependent variables are always in linear relations one to another. A compensator of the transfer function as (8-1) is in the forward path or in the feedback path alone, the coefficients a's and b's are always linear

$$G_c = \frac{s^n + a_{n-1}s^{n-1} + \cdots + a_o}{s^m + b_{m-1}s^{m-1} + \cdots + b_o} \tag{8-1}$$

relations in the partitioned formulas. Those linear algebraic equations are always consistant and therefore always have a solution. For the case in which compensators are in both forward and feedback paths, a quadratic expression is always obtained. But this quadratic expression does not imply the dominant and realizable regions are smaller. It shows only the natural algebraic relations. For example, the coefficients of the reduced characteristic are always real no matter how complicated their functions are.

8-3 Extensions.

The method can be extended to sampled data systems and adaptive systems. The modifications are the different interpretations of the variables. For example, in the case of sampled data systems, if Z transform is used then the criteria of stability, dominance and realizability are different. However, the general idea of approach can be applied.

In the previous chapters, the arbitrary variables are chosen as the roots of the system. This choice is not necessary. Theoretically, any

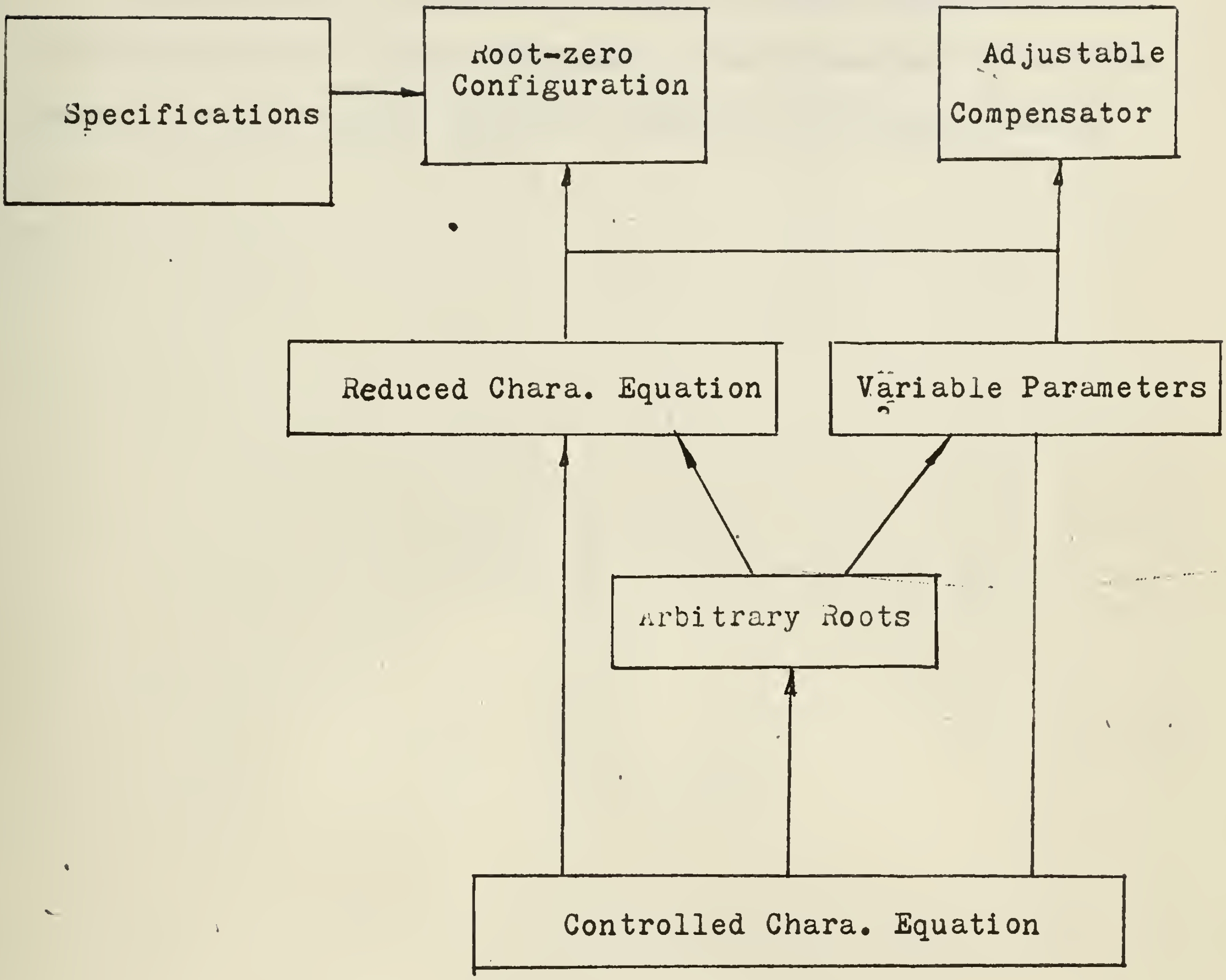

Fig. 8-I General Description of the Method

combination of "r" variables can be chosen as the arbitrary variables such as a combination of roots and variable parameters.

This method can be applied to any mode of linear operation in discontinuous operation. By proper choice of the compensator, the dominant roots of two modes can be chosen in a relation as the designer wants.

APPENDIX I

CONSISTENCY & LINEAR DEPENDENCE OF DERIVATIVE FEEDBACK

1. Introduction.

The coefficients of the characteristic equation of a linear control system are functions of the system parameters. If all the coefficients can be adjusted by a set of the parameters, then the roots of the system can be controlled. Conversely, if the roots are chosen, then the system parameters are forced to have certain values. The conditions of using system parameters to satisfy the coefficients of a characteristic equation for a given set of roots depend upon the number of adjustable parameters and the consistency of the simultaneous equations. Theoretically, if all the derivatives of a system can be obtained from the output of the system, then by feedback as shown in Fig. A-1-1, the system roots can be chosen arbitrarily providing no restrictions are placed on the forward gain K and the control parameters h's. This can be shown as follows. The characteristic equation of the system with feedback is:

$$S^n + (K_2 h_{n-1} + a_{n-1}) S^{n-1} + \cdots + (K_2 h_1 + a_1) S + K_1 K_2 = 0 \quad \text{(A-1-1)}$$

Assume the roots of the system are chosen from the considerations of the specifications, and the required characteristic equation is formulated as equation (A-1-2)

$$S^n + A_{n-1} S^{n-1} + \cdots + A_2 S^2 + A_1 S + A_0 = 0 \quad \text{(A-1-2)}$$

For the system to have the chosen roots, then the corresponding coefficients of equations (A-1-1) and (A-1-2) must be equal. Then

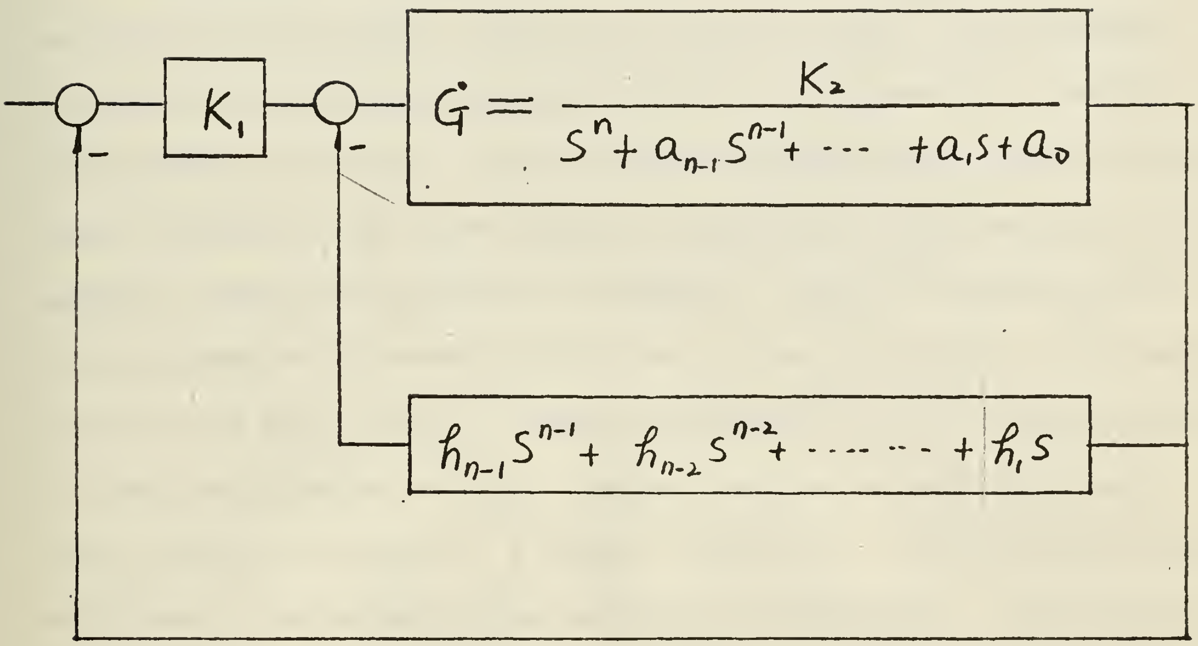

Fig. A-1-1 Block Diagram by Feedback of Derivatives. G is Transfer Function of the Plant. K_1 and K_2 are the System Gains.

$$K_1 K_2 = A_0$$
$$K_2 h_1 + a_1 = A_1 \qquad \text{(A-1-3)}$$
$$\vdots$$
$$K_2 h_{n-1} + a_{n-1} = A_{n-1}$$

Equation (A-1-3) always has a unique solution if there are no restrictions on K's and h's. And in the particular case, each parameter h controls one coefficient independently and the forward gain controls the constant co-efficient. If the control parameters have certain restrictions, then the system roots cannot be arbitrarily chosen, but have to be subject to these restrictions. For example, if K_1K_2, is specified from the consideration of steady state accuracy, then the product of the roots must equal to K_1K_2. If h_{n-1} cannot be obtained ($h_{n-1} = 0$), then the sum of roots must be equal to a_{n-1}. The restrictions on the control parameters restrict the roots in a certain constraint. The more restrictions, the higher is the degree of the constraints of the roots. In a practical system, the higher derivatives are very noisy and very difficult to obtain. This is a common restriction for derivative feedback.

In Fig. A-1-1, it was assumed that there were no zeros in the forward path. If one zero is in the forward path, then the allowed highest order of the feedback derivative is (n-2). In general, if "m" zeros are in the forward path, only the derivatives of the order up to n-(M+1) are allowed to feedback. This can be proved as follows

Refer to Fig. A-1-1, assume m zeros in G_T , and the highest derivative has an order of r . Then the characteristic equation is

$$s^n + (a_{n-1} + h_{r+m})\, s^{n-1} + \cdots = 0 \qquad \text{(A-1-4)}$$

The subscript ($r+m$) of h_{r+m} indicates the highest order of the feedback derivative and the number of zeros. From equation (A-1-4)

$$r+m < n-1$$

$$r < n-(m+1)$$

Therefore if m zeros are in the forward path, then only n-($m+1$) control parameters can be introduced, if the feedback path is as shown in Fig. A-1-1. However, other schemes of lower order derivative feedback are possible to introduce (n-1) control parameters and consequently control all the coefficients of the characteristic equation. The following sections are discussions about these possibilities. In the following sections, two terminologies are defined as follows:

Control variable is the variable of the system which can be measured.

Control parameter is the adjustable parameter in the system.

2. Feedback of the First and Second Derivative from the Output to Several Points in the Forward Path.

Since the first and the second derivative are usually allowed to feedback for high order systems and can be obtained, only these two derivatives are considered. The object of this discussion is to adjust all the coefficients of a nth order characteristic equation to desired values. In order to accomplish this, n control parameters are needed. As the forward gain K is considered as one control parameter, (n-1) control parameters from the feedback path are necessary for this particular scheme. Using first and second derivative feedback around the plant intorduces only two control parameters. However, if there exist junction points between the energy-storage elements in the forward path, such that signals can be fed in or picked up, then additional control parameters can be introduced by feeding derivatives to those points. As two control parameters have been introduced by feeding derivatives to the error channel, then the necessary number of points between the energy storage elements in the forward path is $\frac{n-3}{2}$ (if n is cdd and $\frac{n-2}{2}$ if n is even) since two derivatives can be fed into each node. Briefly, this can be summarized in the following two statements.

(1)* If a nth order system has at least $\frac{n-3}{2}$ available points (if n is odd, and $\frac{n-2}{2}$ if n is even) between the energy storage elements in the forward path, then all the coefficients of the characteristic equation can be adjusted to arbitrary values by introducing first and second derivatives from the output to each point providing the two derivatives are defined and the feedback paths are proper.

*all statements in this discussion may be considered as rules. All of them are necessary conditions, but not sufficient conditions.

(2) If a nth order system has at least n-2 available points between the energy-storage elements in the forward path, then all the coefficients of the characteristic equation can be adjusted to arbitrary values by introducing only first derivative feedback from the output to each point, providing the feedback paths are proper.

3. Formulation of Characteristic Equation for Pure Derivative Feedback.

From the signal flow graph, the determinant of a system can be obtained by the following rule.

$$\Delta = 1 - \sum_K L_K^{(1)} + \sum_K L_K^{(2)}$$

where $L_K^{(n)}$ = product of the kth possible combination of n non-touching loops

L = loop transmission

For this particular derivative feedback system, two terminologies are defined as follows:

Loop gain = numerator of each loop transmission.

In Fig. A-1-2. Loop gain of $L_1 = h_1 P_3$

Loop gain of $L_2 = h_{2-2} P_3$

Loop gain of $L_3 = h_3 P_1 P_2 P_3$

Loop gain of $L_4 = P_1 P_2 P_3$

Characteristic function = denominator of each of the forward transfer functions, or the product of them.

Consider Fig. A-1-2: from the graph rule

$$\Delta = 1 + L_1 + L_2 + L_3 + L_4$$

$$= 1 + \frac{P_3 h_1}{Q_3} + \frac{P_2 P_3 h_2}{Q_2 Q_3} + \frac{P_1 P_2 P_3 h_3}{Q_1 Q_2 Q_3} + \frac{P_1 P_2 P_2}{Q_1 Q_2 Q_3}$$

$$= Q_1 Q_2 Q_3 + Q_1 Q_2 (P_3 h_1) + Q_1 (P_2 P_3 h_2) + P_1 P_2 P_3 h_3 + P_1 P_2 P_3 \qquad \text{(A-1-5)}$$

The first term $Q_1 Q_2 Q_3$ + last term $P_1 P_2 P_3$ = characteristic equation of the original system.

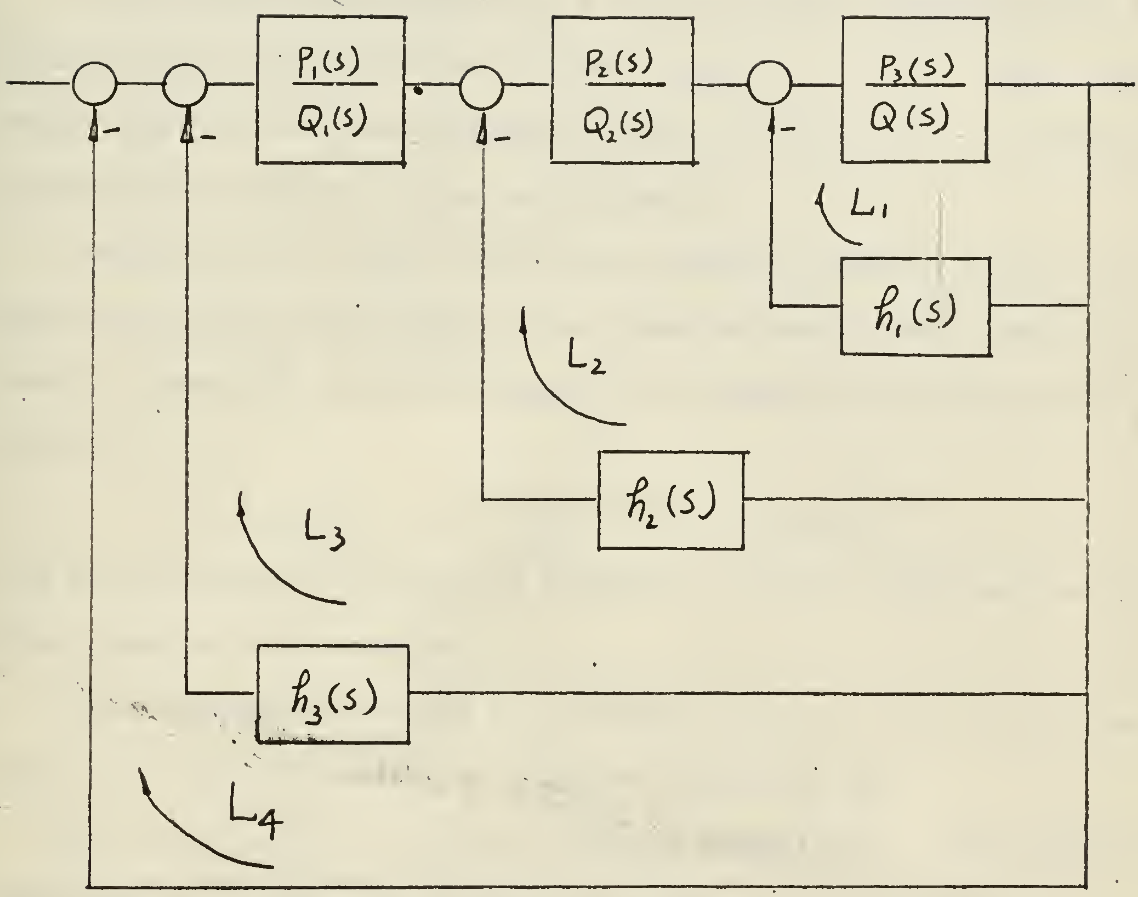

Fig. A-1-2 Formulation of the Determinant

$P_i(s)$, $Q_i(s)$, $h_i(s)$ are polynominal in s

The second term = loop gain L_1 X nontouching chara. function.

The third term = loop gain L_2 X nontouching chara. function.

The fourth term = loop gain L_3 X nontouching chara. function.

From equation (A-1-5), the formulation of the characteristic equation for pure derivative feedback may be stated in the following rule.

The characteristic equation of a pure derivative feedback system = characteristic equation of the original system + $\sum_{K}^{(1)}$ loop gain X characteristic function of nontouching loop + $\sum_{K}^{(2)}$ loop gain X characteristic function on nontouching loop.

Example A-1-1: Devise a derivative feedback scheme of Fig. A-1-3, such that all the coefficients of the characteristic equation can be adjusted. Assume the required characteristic equation from the specifications is

$$S^5 + A_4 S^4 + A_3 S^3 + A_2 S^2 + A_1 S + A_0 = 0$$

find the relations of the control parameters and the coefficients of the given characteristic equation.

Since intermediate points are available, then n-2 = 5-2 = 3. From rule (2), it is not possible to use first derivative only.

By rule (1) $\frac{n-3}{2} = \frac{5-3}{2} = 1$. Then a combination of first and second derivative can meet the requirement. There are many possible ways to accomplish this since only 4 control parameters are needed.

Two schemes are listed as follows:

(a) $H_1 = h_1 s \qquad H_2 = h_2 s^2 + h_3 s \qquad H_3 = h_4 s^2$

(b) $H_1 = h_1 s \qquad H_2 = h_2 s^2 \qquad H_3 = h_3 s^2 + h_4 s$

Assume scheme (a) is chosen. The chara. equation is formulated in a tabulated form for convenience.

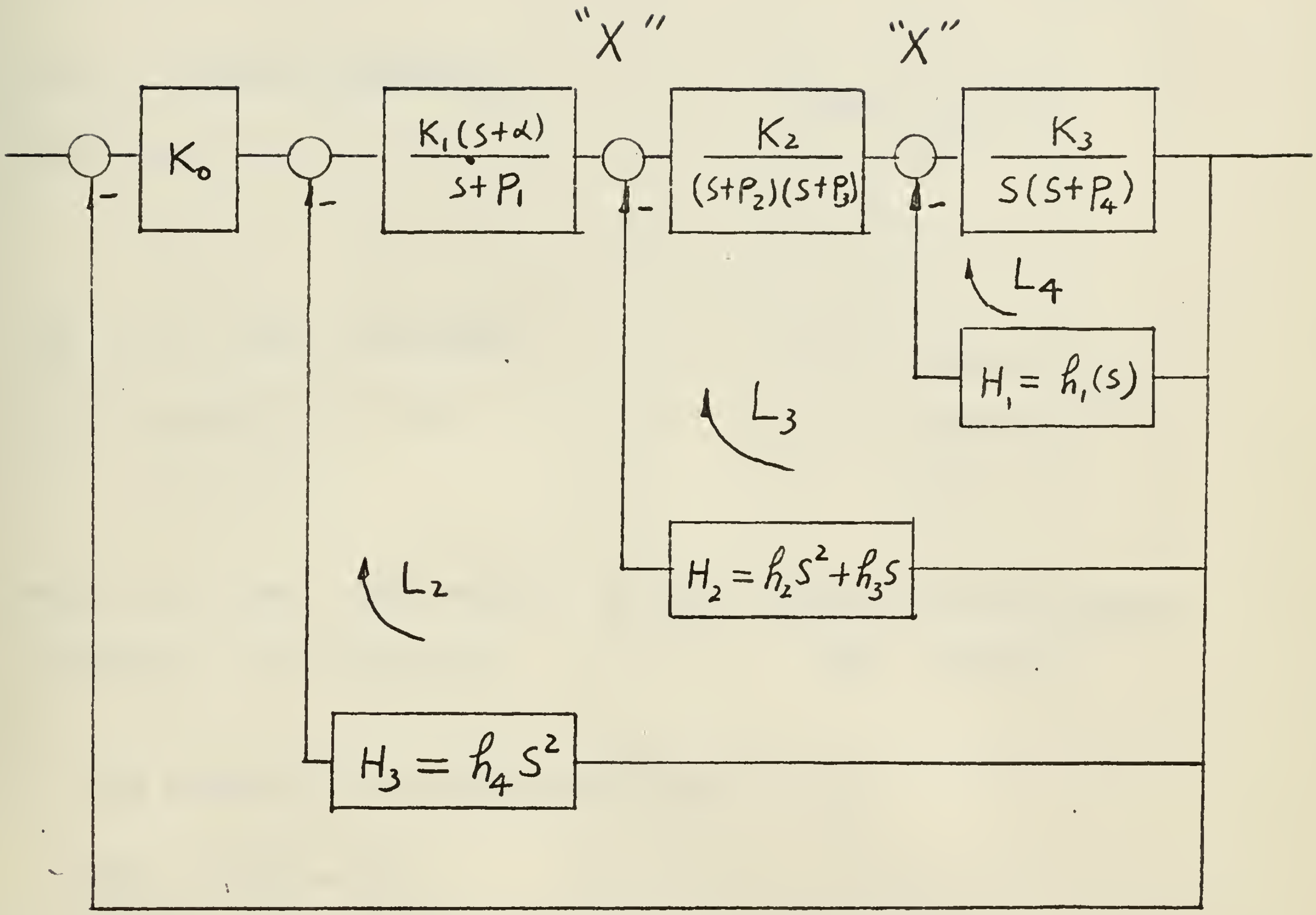

Fig. A-1-3. Example A-1-1 - Mark "X" denotes signals are allowed to feed in or pick up

	s^5	$A_4 s^4$	$A_3 s^3$	$A_2 s^2$	$A_1 s$	A_0
Required Chara. Equation						
Original Chara. Equation	1	ΣP_i	$\Sigma P_i P_j$	$\Sigma P_i P_j P_K$	$\prod\limits_i P_i + K_0K_1K_2K_3$	$\alpha K_0K_1K_2K_3$
Gain L_2 x Chara. Nontouching $h_4 s^2 K_1K_2K_3\,(s+\alpha)$			$K_1K_2K_3h_4$	$K_1K_2K_3\alpha h_4$		
Gain L_3 x Chara. Nontouching $(h_2s^2+h_3s)K_2K_3 \times (s+P_1)$			$K_2K_3h_2$	$K_2K_3P_1h_2 + K_2K_3h_3$	$K_2K_3P_1h_3$	
Gain L_4 x Chara. Func. Nontou. $h_1 s K_3 \times (s+P_1)(s+P_2)(s+P_3)$		K_3h_1	$K_3h_1(P_1+P_2+P_3)$	$K_3(P_1P_2+P_2P_3+P_3P_1)h_1$	$K_3P_1P_2P_3h_1$	

The equations for the coefficients are

$$K_3h_1 + \Sigma P_i = A_4$$
$$\Sigma P_iP_j + K_1K_2K_3h_4 + K_2K_3h_2 + K_3(P_1+P_2+P_3)h_1 = A_3$$
$$\Sigma P_iP_jP_K + K_1K_2K_3\alpha h_4 + K_2K_3P_1h_2 + K_2K_3h_3 + K_3(P_1P_2+P_2P_3+P_3P_1)h_1 = A_2$$
$$\prod_i P_i + K_0K_1K_2K_3 + K_2K_3P_1h_2 + K_3P_1P_2P_3h_1 = A_1$$
$$\alpha K_0K_1K_2K_3 = A_0$$

The first four equations of (A-1-6) can be written in matrix form:

$$\begin{bmatrix} K_3 & 0 & 0 & 0 \\ K_3(P_1+P_2+P_3) & K_2K_3 & 0 & K_1K_2K_3 \\ K_3(P_1P_2+P_2P_3+P_3P_1) & K_2K_3P_1 & K_2K_3 & K_1K_2K_3\alpha \\ K_3P_1P_2P_3 & 0 & K_2K_3P_1 & 0 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix} = \begin{bmatrix} A_4 - \Sigma P_i \\ A_3 - \Sigma P_iP_j \\ A_2 - \Sigma P_iP_jP_K \\ A_1 - \prod P_i - \prod\limits_{i=0}^{3} P_i \end{bmatrix}$$

In (A-1-7) whenever the determinant is not zero, there always exists a solution of the control parameters h's for arbitrary values of A.

In the above example, the table of the characteristic equations gives a clear picture of how the coefficients are controlled by h-parameters. Also equation (A-1-7) shows that the h-parameters are linearly dependent to the coefficients of the given characteristic equation. It is this linear dependence which makes the arbitrary adjustment possible. In equation (A-1-6), the constant term is controlled by the forward gain only. This is the nature of derivative feedback, because the derivatives control the coefficient of the terms of a differential equation. Any derivative feedback from the output has the scheme of properties of linear dependence of h-parameters and the independence of forward gain K.

Example (A-1-2). Fig. A-1-4 is a third order system, one point between the energy-storage elements in the forward path is allowed to feed in signal, then the system roots may be arbitrarily chosen. Assume the roots to be -3 and -0.5 $\pm$ j 1. K_1, h_1, h_2 can be determined. The characteristic equation for the given roots is;

	S^3	+	$4S^2$	+	$4.25S$	+	$3.75 = 0$
Main loop	1		1		0		K_1
L_1: loop gain = K_1h_2s chara. function not enclosed = 1					K_1h_2		
L_2: loop gain = h_1s Chara. function not enclosed = s + 1			h_1		h_1		

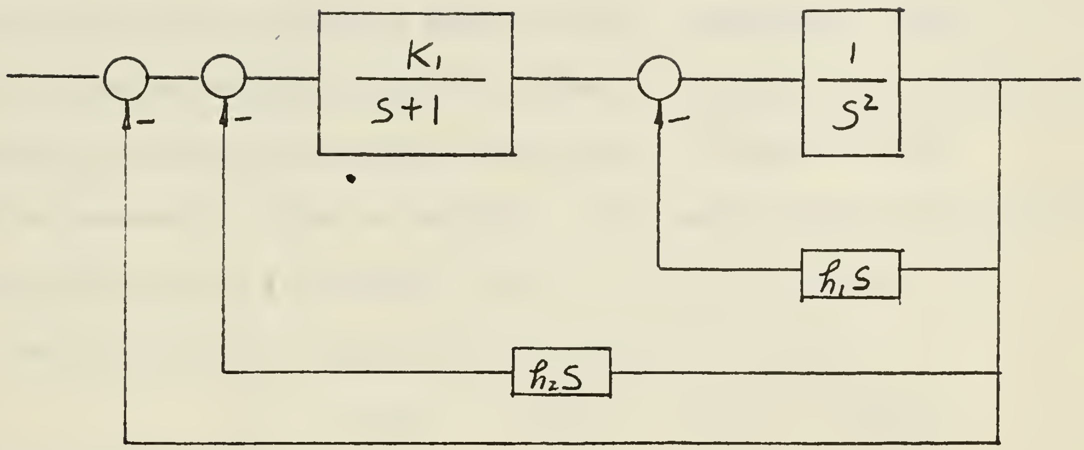

Fig. A-I-4 Example A-I-2

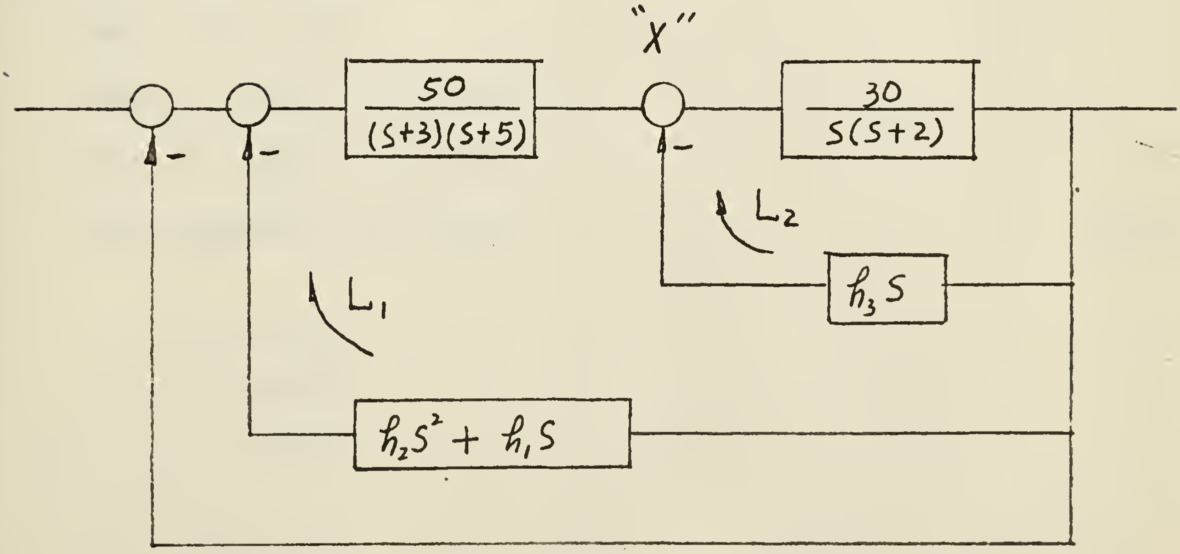

Fig. A-I-5 Example A-I-3

Set the corresponding coefficients equal, variable parameters are solved.

$$K_1 = 3.75, \qquad h_1 = 3, \qquad h_2 = 0.334$$

Example A-1-3. In Fig. A-1-5, it is assumed that signal can be fed in at the point "X" in the forward path, and the forward gain of the system is 1500 based on the steady state specification. According to rule 1, $\frac{n-2}{2} = 1$, then by first and second derivative feedback, the system roots can be chosen arbitrarily providing the product of the roots is equal to 1500.

The uncompensated system is unstable. The feedback paths are chosen as shown, and the roots are assumed to be $S = (-3 \pm j5)$, -6, and -7.36.

The characteristic equation for the chosen roots is

$$S^4 + 19.36S^3 + 158.5^2 + 720S + 1500 = 0$$

The system coefficients

Uncompensated system =	1	10	31	30	1500
L_1: $(h_1 s + h_2 s^2)\ 1500$			$1500h_2$	$1500h_1$	
L_2: $h_3 s\ 30(s+3)(s+5)$		$30h_3$	$240h_3$	$450h_3$	

The coefficients relations are:

$$\begin{aligned} 30h_3 + 10 &= 19.36 \\ 1500h_2 + 240h_3 + 31 &= 158 \\ 450h_3 + 1500h_1 + 30 &= 720 \end{aligned} \tag{A-1-8}$$

Solve equation (A-1-8), get

$$h_1 = 0.367$$

$$h_2 = 0.0347$$

$$h_3 = 0.312$$

5. Feedback from Intermediate Points.

It can be shown that if the output of each block in the diagram (A-1-6) is chosen as the state variable of the system, then there exists a linear transforamtion between the state space and the phase space (output and all its derivatives) of a system. In Fig. A-1-6, the state variables are chosen as shown, the relationship between state space and phase space is

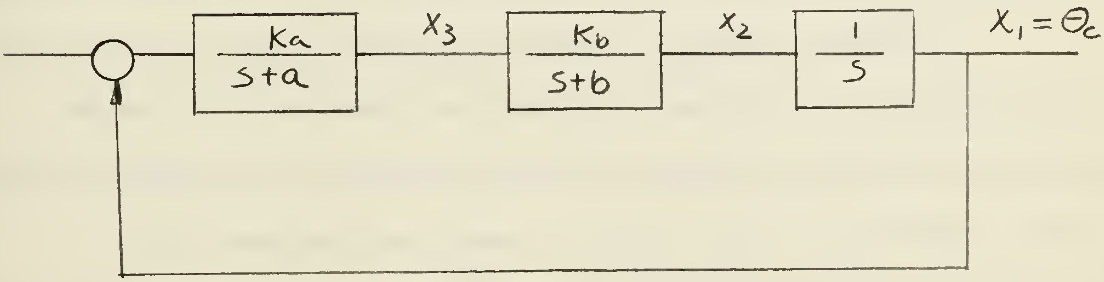

Fig. A-1-6 State Variables

derived as follows:

$$x_1 = \theta_c \qquad \text{(A-1-9-1)}$$

$$\frac{x_1}{x_2} = \frac{1}{s}, \qquad x_2 = \dot{x}_1 = \frac{d\theta}{dt} \qquad \text{(A-1-9-2)}$$

$$\frac{x_2}{x_3} = \frac{K_b}{s+b}, \quad \dot{x}_2 + b x_2 = K_b x_3, \quad x_3 = \frac{b}{K_b}\left(\frac{d\theta}{dt}\right) + \frac{1}{K_b}\left(\frac{d^2\theta}{dt^2}\right) \qquad \text{(A-1-9-3)}$$

Equations (A-1-9) are the required relations. It shows that the output of each block is a linear combination of the output θ and its derivatives.

With this concept, therefore, the feedback from intermediate points is equivalent to derivative feedback from the output. Fig. A-1-7(a) is equivalent to Fig. A-1-7(b).

This equivalence also can be shown by block diagram manipulations. Because of this equivalence, if the signals flowing in the intermediate points can be measured, then those signals can be taken as control variables and the analysis is the same as for derivative feedback from the output. Moreover, if the derivatives of those signals are defined and can be obtained, they also can be taken as control variables.

However, if consistency and linear dependence are considered, the feedback paths and the number of control parameters have many restrictions.

(1) Consider the scheme of Fig. A-1-8, it introduces sufficient control parameters but the coefficients of the characteristic equation are not linear combinations of the h-parameters. It consists of two independent nontouching loops, the contribution of these two loops to the whole system is the product of the two loop transmissions. So the coefficients are quadratic functions of the h- parameters. This can be shown by formulating the characteristic equation as follows:

	S^3	S^2	S^1	S^o
Original system	1	$a+b$	ab	$K_a K_b$
L_1 : $K_b h_1 S(S+a)$		$K_b h_1$	$K_a a h_1$	
L_2 $K_a h_2 (S^2+bS)$		$K_a h_2$	$K_a b h_2$	
$L_1 \times L_2$			$K_a K_b h_1 h_2$	

If the required characteristic equation is

$$S^3 + A_2 S^2 + A_1 S + A_o = 0$$

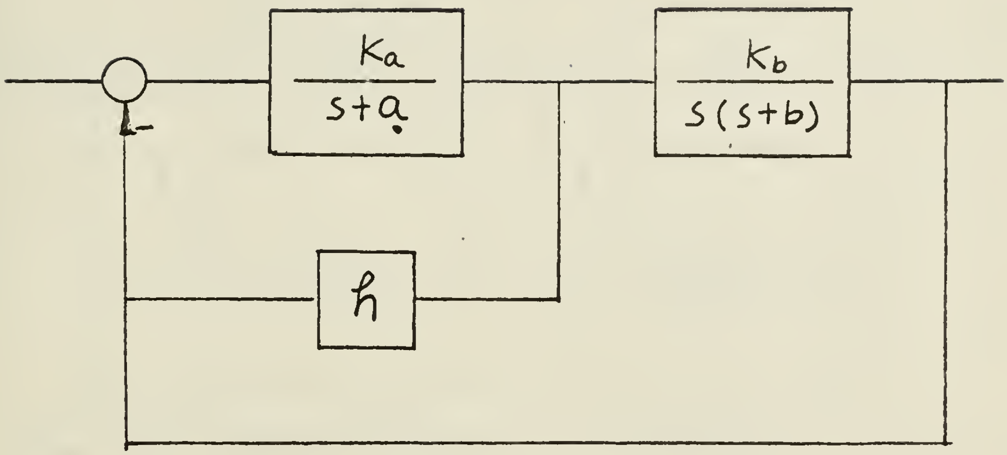

Fig. A-1-7(a) Feedback from Intermediate points.

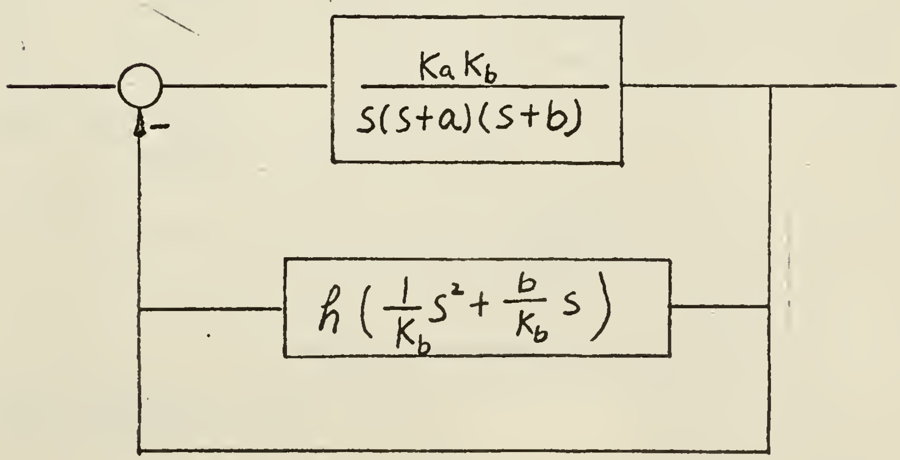

Fig. A-1-7(b) Equivalenence of Fig. A-1-7(a)

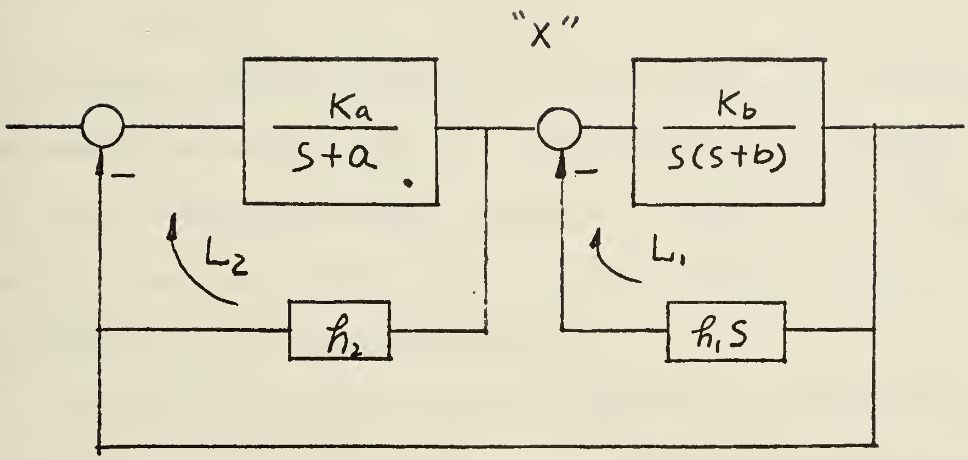

Fig. A-1-8 Quardrtic Function of h parameters

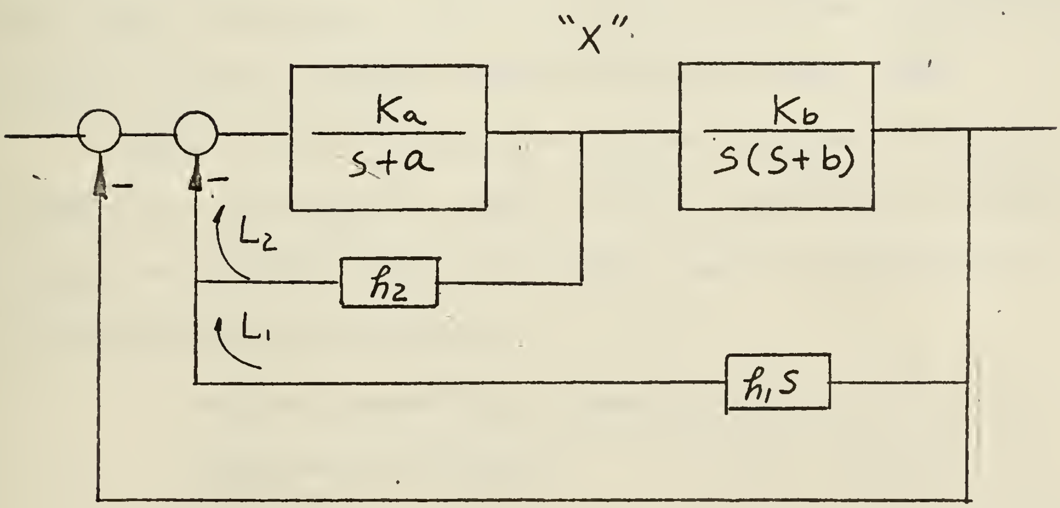

Fig. A-1-9 Example A-1-4

then

$$K_b h_1 + K_a h_2 = A_2 - (a+b)$$

$$K_a a h_1 + K_a b h_2 + K_a K_b h_1 h_2 = A_1 - ab \qquad \text{(A-1-10)}$$

$$K_a K_b = A_o$$

The solution of equation (A-1-10) for real values of h_1 and H_2 implies the restriction of the roots in a certain region. In general, if there are " r " independent feedback loops in a system, the coefficients of the characteristic equation consist of the terms h_1, $(h_1 h_2)$, ... $(h_1 ... h_r)$. This leads to the following statement.

If all the coefficients of the characteristic equation of a derivative controlled system are a linear function of the control parameter, then all feedback paths must be interconnected, (or touching). The above statement may be stated in another way: If all the coefficients of the characteristic equation of a derivative controlled system be a linear function of the control parameters, then those points from which the signals are taken as control variables cannot be used as feed-in nodes.

Example A-1-4. In Fig. A-1-9, assume the signal flowing at the point marked "X" can be measured, then it can be picked up as a control variable. Assume the feedback paths are as shown, they are interconnected. The characteristic equations are:

	S^3	$+ \; A_2 S^2$	$+ \; A_1 S$	$+ \; A_o = 0$
Required characteristic equation =	S^3	$A_2 S^2$	$A_1 S$	A_o
Characteristic equation	1	$a+b$	ab	$K_a K_b$
$L_1 = h_1 s K_a K_b$			$K_a K_b h_1$	
$L_2 = h_2 K_a (s^2 + bs)$		$K_a h_2$	$K_a h_2 b$	

The coefficient relations are:

$$K_a h_2 = A_2 - (a+b)$$

$$K_a K_b h_1 + K_a b h_2 = A_1 - ab \qquad \text{(A-1-11)}$$

$$K_a K_b = A_o$$

h_1 and h_2 can be solved for arbitrary value of A's.

(2) If zeros are involved in the forward path, the feedback from the intermediate nodes may introduce the problem of consistency. Consider Fig. A-1-10, the signal at "X" can be measured and then can be taken as control variable. The characteristic equations are:

Required Chara. Equation	S^3 +	A_2S^2 +	A_1S +	$A_o = 0$
Original loop	1	$a + b$	$ab+k_ak_b$	K_aK_b
L_2 : $h_2 S K_a K_b (S+\alpha)$		$K_aK_bh_2$	$K_aK_b\alpha h_2$	
L_1 : $K_a(S+\alpha)(S^2+bS)h_1$	K_ah_1	$K_a(S+b)h_1$	$K_a\alpha b h_1$	

The coefficient relations are:

$$\begin{aligned} & h_a h_1 = 1 \\ & K_aK_bh_2 + K_a(\alpha+b)h_1 = A_2 - (a+b) \\ & K_a\alpha b h_1 + K_aK_b\alpha h_2 = A_1 - (ab + K_aK_b) \\ & K_aK_b\alpha = A_o \end{aligned} \qquad \text{(A-1-12)}$$

Equation (A-1-12) are linear, but are not consistent for arbitrary A's, since the number of equations is one more than the number of unknowns (h_1 and h_2). Substitute h_1 from the first equation into the second and third equation, obtain:

$$K_aK_bh_2 + (\alpha + b) = A_2 - (a + b)$$

$$K_aK_b\alpha h_2 + \alpha b = A_1 - (ab + K_aK_b)$$

Manipulate, get

$$\alpha A_2 - A_1 = \alpha(\alpha + a - 3b) - ab - K_aK_b \qquad \text{(A-1-13)}$$

Equation (A-1-13) is the restriction of the roots for this systm, i.e., if equation (A-1-12) has a solution, then equation (A-1-13) must be satisfied. Therefore, when a system has the required number of control parameters and

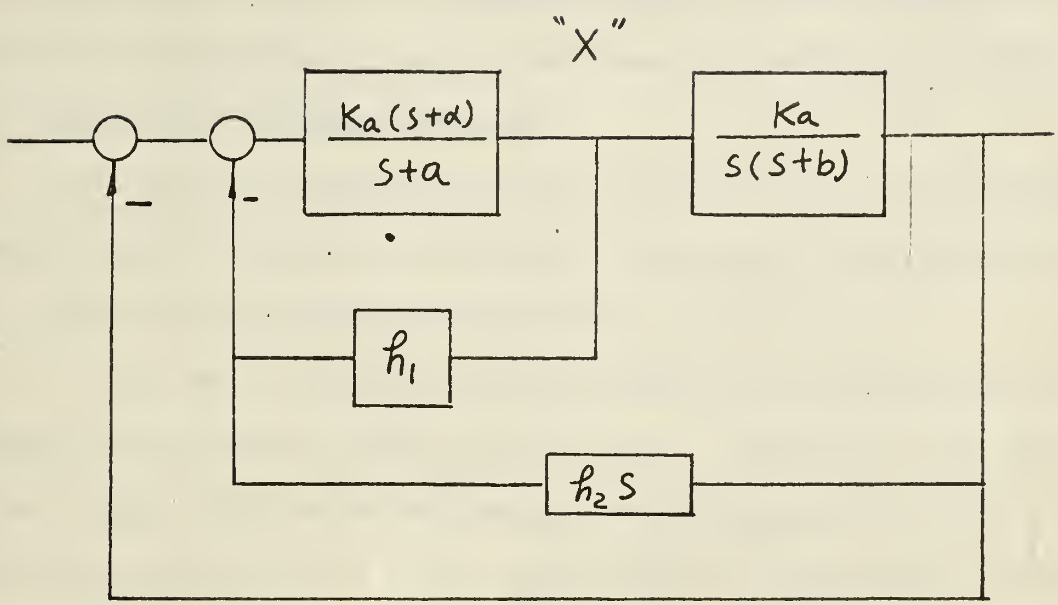

Fig. A-1-10. Effect of Zeros to Feedback

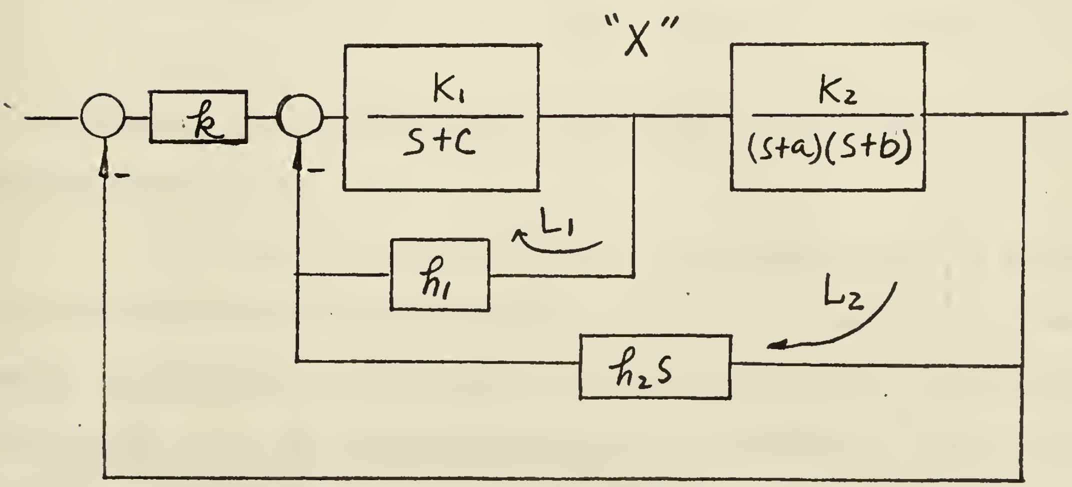

Fig. A-1-11. Feedback of the output to more than one point.

the linear relationship of the coefficients, the choice of roots may still be subject to some restriction if the feedback is not proper. In general it can be shown that if the loop transmission has the same order(s) for both numerator and denominator, then it introduces the restriction of the roots. This leads to the following statement.

If the roots of derivative controlled systems are to be chosen arbitrarily, then for each loop transmission, the order of the numerator must be at least one less than the denominator.

(3) There is another obvious inconsistency which arises from feedback of the output to more than one point. Therefore, if an intermediate signal which is not followed by pure integrator, be fed to another point, the equations of the coefficients is inconsistent for arbitrary choice of roots. Fig. A-1-11 is an example.

The characteristic equation of Fig. A-1-11 is

	s^3	s^2	s^1	s^0
original loop	1	$a+b+c$	$ab+bc+ba$	K_1K_2k+abc
L_1: $h_1K_1(s+a)(s+b)$		K_1h_1	$(a+b)K_1h_1$	abK_1h_1
L_2: $h_2K_1K_2s$			$K_1K_2h_2$	

There are three equations but two unknowns. This leads to a constraint of the roots.

(4) Since the feedback from an intermediate node is equivalent to derivative feedback from the output, then it is possible that the parameters introduced by both control variables may have the same effect on the system; i.e., the parameters may not be independent. This can be easily detected by just looking at the loops. In Fig. A-1-12, h_3 has the same effect as h_2 and h_1, since h_3 is a combination of the first and second derivative of the output. That is, h_3 enters the coefficient

of the first and second derivative terms as h_1 and h_2. Therefore in Fig. A-1-12, the system actually has only two control parameters.

However, if the derivative of the signal at "X" can be obtained and the scheme changes to Fig. A-1-13, then each parameter plays its own role.

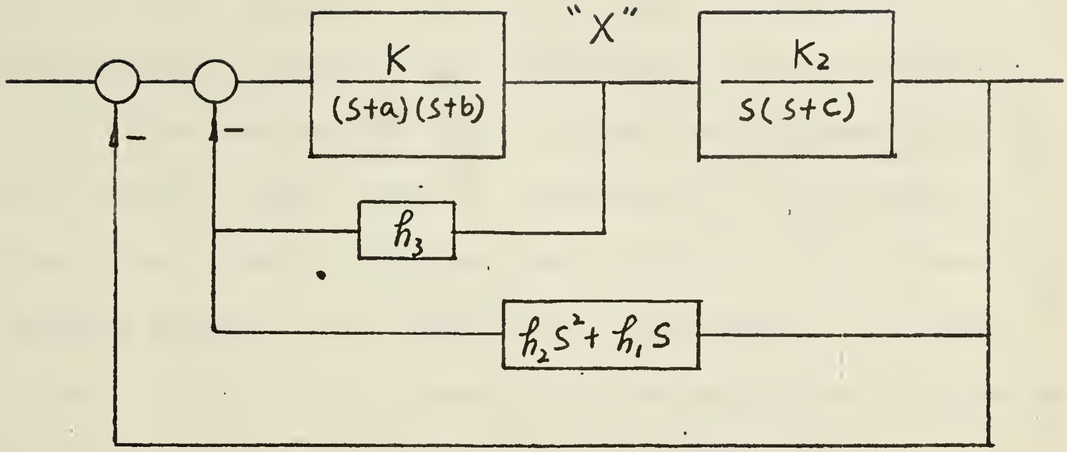

Fig. A-1-12. Two parameters have the same effect.

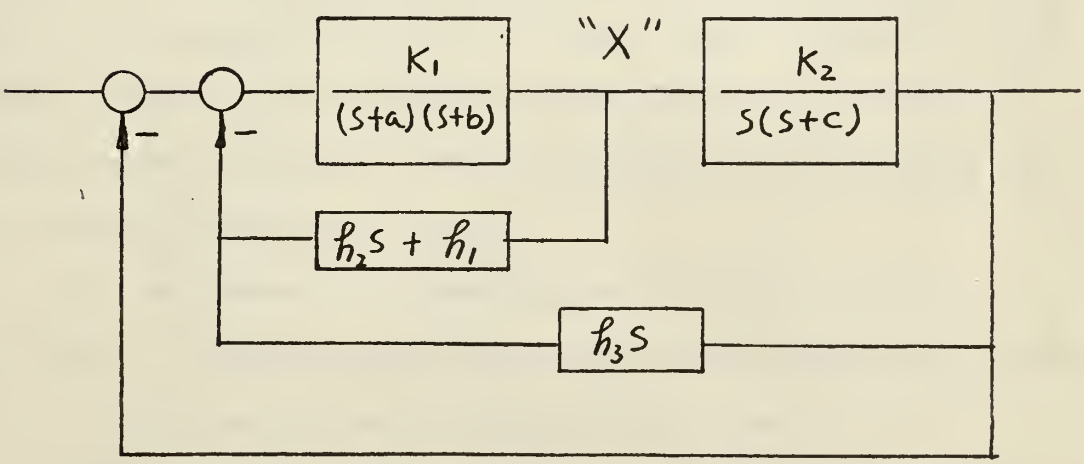

Fig. A-1-13. Each parameter has it's own effect

5. Summary.

Derivative control is an effective method to control the coefficients of the characteristic equation of a system, since it uses derivatives of the system variables to control the coefficients of the characteristic equation. Since an intermediate signal feedback has the same effect as derivatives from the output, they are considered as in the category of derivatives. The criteria in Section I also applies to the intermediate signal feedback because of the condition (1) in Section 4. The intermediate signals do not introduce more control parameters but increase the possible ways of obtaining derivatives and the feedback path.

In general, one constraint of the roots is almost always required in a system,that is, the product of the roots equal to the system forward gain, because the gain is usually specified from the stiffness of the system. This is only one degree of constraint, it does not introduce much difficulty. But if additional constraints are involved such as quadratic functions and inconsistent equations, then the solution is not so obvious. Therefore, it is logical to look for a set of linear and consistent control parameters. From the discussion in the previous sections, it may be summarized as follows:

(1) The number of h parameters must be equal to (n-1)

(2) h - parameters must appear only in the coefficients from the first derivatives to the (n-1) derivative.

(3) h-parameters must be linear combinations of the given coefficients.

Example A-1-5. Fig. 2-1-14 is a 6th order system. Assume two nodes between the energy-storage element are allowed to feed in or pickup signals, and the derivative of the signal at "X"$_1$ can be obtained. From the

criteria of intermediate nodes, ($\frac{6-2}{2}$ = 2) the system coefficients can be adjusted to arbitrary value by first and second derivative feedback. Fig. A-1-14 (a) (b) and (c) are the possible proper feedback paths. Fig. A-1-14(c) again has 5 possible combinations of H's. They are

$H_1 = h_1 s + h_2$	$H_1 = h_1 s$ (or h_1)	$H_1 = h_1 s + h_2$
$H_2 = h_3 s + h_4$	$H_2 = h_2 s + h_3$	$H_2 = h_3 s$
$H_3 = h_5 s$ (or $h_5 s^2$)	$H_3 = h_4 s^2 + h_5 s$	$H_3 = h_4 s^2 + h_5 s$

Therefore, there are 8 possible ways to control the coefficients of the characteristic equation of this system. The choice of the scheme depends upon the specifications and perhaps the engineering judgments.

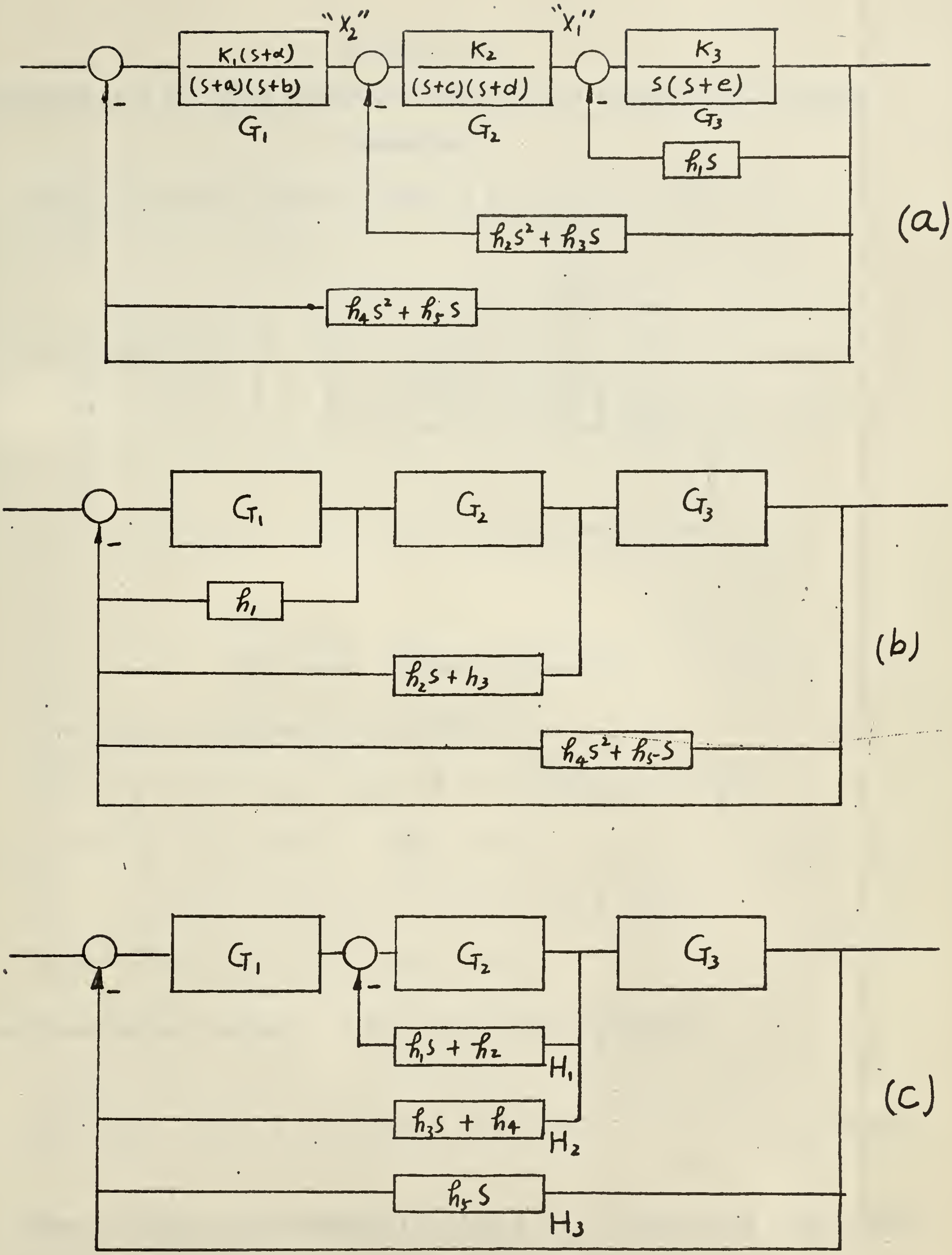

Fig. A-1-14. Example of A-1-5.

APPENDIX II

PARTITION OF A nTH ORDER SYSTEM WITH ONE ZERO FOR SINGLE SECTION CASCADE COMPENSATOR

Take a 4th order system as shown in Fig. A-2-1.

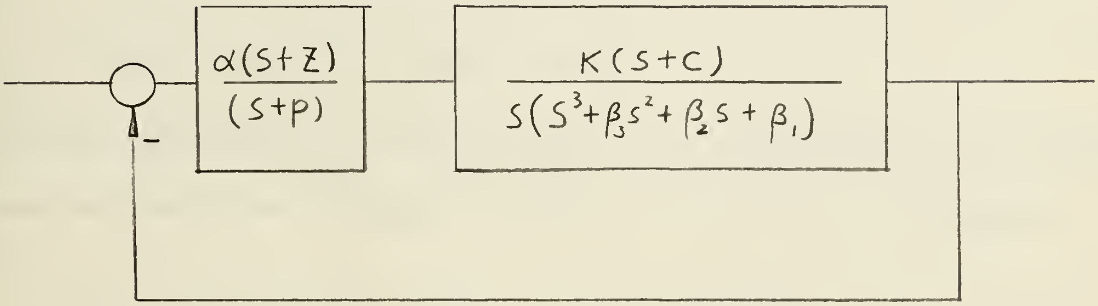

Fig. A-2-1. Cascade Compensator

The controlled characteristic equation is

$$S^5 + (\beta_3 + p)S^4 + (\beta_2 + \beta p)S^3 + (\beta_1 + P\beta_2)S^2 + \beta_1 P S$$
$$+ K\alpha S^2 + K\alpha(z + c)S + KpC = 0$$

$$S^5 + (\beta_3 + P)S^4 + (\beta_2 + \beta_3 P)S^3 + (\beta_1 + P\beta_2 + K\alpha)S^2$$
$$+ (\beta_1 P + KP + K\alpha C)S + KPC = 0 \qquad \text{(A-2-1)}$$

The characteristic equation of the uncompensated system is

$$S^4 + \beta_3 S^3 + \beta_2 S^2 + (\beta_1 + K)S + KC = 0 \qquad \text{(A-2-2)}$$

In order to express the coefficients of (A-2-1) in terms of the coefficients of the uncompensated system in a simpler form, define:

$$
\begin{aligned}
\beta_3 &\triangleq B_3 \\
\beta_2 &\triangleq B_2 \\
\beta_1 + K &\triangleq B_1 \\
KC &\triangleq B_0
\end{aligned}
\qquad \text{(A-2-3)}
$$

Substitute (A-2-3) into (A-2-1) and manipulate, one obtains

$$
\begin{aligned}
&S^5 + (B_3 + p)S^4 + (B_2 + pB_3)S^3 + \left(B_1 - \frac{B_0}{C} + \frac{B_0}{C}\alpha + pB_2\right)S^2 \\
&+ (pB_1 + \alpha B_0)S + B_0 p = 0
\end{aligned}
\qquad \text{(A-2-4)}
$$

Set the coefficients of (A-2-4) equal to the corresponding root-coefficients and denote the arbitrary roots by ρ and ω_n, one obtains

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & -1 \\
B_3 & 0 & 0 & -1 & -2\rho\omega_n \\
B_2 & \frac{B_0}{C} & -1 & -2\rho\omega_n & -\omega_n^2 \\
B_1 & B_0 & -2\rho\omega_n & -\omega_n^2 & 0 \\
B_0 & 0 & -\omega_n^2 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
p \\ \alpha \\ C_0 \\ C_1 \\ C_2
\end{bmatrix}
=
\begin{bmatrix}
2\rho\omega_n - B_3 \\
\omega_n^2 - B_2 \\
-B_1 + \frac{B_0}{C} \\
0 \\
0
\end{bmatrix}
\qquad \text{(A-2-5)}
$$

Where C_0, C_1 and C_2 are the coefficients of the reduced characteristic equation.

By Cramer's Rule.

$$
C_0 = \frac{B_0\left[B_1 - 2\rho\omega_n B_2 - (1-4\rho^2)B_3\omega_n^2 - (-4\rho + 8\rho^3)\omega_n^3\right] - \frac{B_0}{C}\left[B_0 - B_2\omega_n^2 + 2\rho\omega_n^3 B_3 + (1-4\rho^2)\omega_n^4\right]}{\left[B_0 - B_2\omega_n^2 + 2\rho B_3\omega_n^3 + (1-4\rho^2)\right]\omega_n^4 + \frac{\omega_n}{C}\left[-2\rho B_0 + B_1\omega_n - B_3\omega_n^3 + 2\rho\omega_n^4\right]}
\qquad \text{(A-2-6)}
$$

Define:

$$
\begin{aligned}
\varphi_0 &= 0 \\
\varphi_1 &= 1 \\
\varphi_2 &= 2\rho \\
\varphi_3 &= 1 - 4\rho^2
\end{aligned}
$$

$$\vdots$$

$$\varphi_K = -\left[2\zeta \varphi_{K-1} + \varphi_{K-2} \right] \quad \text{for} \quad K > 2$$

For an uncompensated nth order system which has one open loop zero at (-C) and has the characteristic equation with coefficients B_{n-1}, B_{n-2}, ...B_o, then

$$C_o = \frac{B_o\left[B_1 - \varphi_2 B_2 \omega_n - \varphi_3 B_3 \omega_n^2 - \varphi_4 B_4 \omega_n^3 - \cdots\right] - \frac{B_o}{C}\left[B_o + \varphi_1 B_2 \omega_n^2 + \varphi_2 B_3 \omega_n^3 + \cdots\right]}{\left[B_o + \varphi_1 B_2 \omega_n^2 + \varphi_2 B_3 \omega_n^3 + \varphi_3 B_4 \omega_n^4 + \cdots\right] + \frac{\omega_n}{C_o}\left[(-\varphi_2 B_o - \varphi_1 B_1 \omega_n) + \varphi_1 \omega_n^3 + \varphi_2 B_4 \omega_n^4 + \cdots\right]} \tag{A-2-7}$$

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